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J. M. Manning

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ELEMENTS OF GEOMETRY,

BY

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TRANSLATED FROM THE FRENCH

FOR

THE USE OF THE STUDENTS OF THE UNIVERSITY

AT

CAMBRIDGE, NEW ENGLAND,

BY JOHN FARRAR,

PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY.

SECOND EDITION,

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It is necessary to the understanding of this work that the reader should have a knowledge of the theory of proportions which is explained in common treatises either of arithmetic or algebra; he is supposed also to be acquainted with the first rules of algebra; such as the addition and subtraction of quantities, and the most simple operations belonging to equations of the first degree. The ancients, who had not a knowledge of algebra, supplied the want of it by reasoning and by the use of proportions which they managed with great dexterity. As for us, who have this instrument in addition to what they possessed, we should do wrong not to make use of it, if any new facilities are to be derived from it. I have accordingly not hesitated to employ the signs and operations of algebra, when I have thought it necessary, but I have guarded against involving in difficult operations what ought by its nature to be simple; and all the use I have made of algebra in these elements, consists as I have already said, in a few very simple rules, which may be understood almost without suspecting that they belong to algebra.

Besides, it has appeared to me, that, if the study of geometry ought to be preceded by certain lessons in algebra, it would be not less advantageous to carry on the study of these two sciences together, and to intermix them as much as possible. According as we advance in geometry, we find it necessary to combine together a greater number of relations, and algebra may be of great service in conducting us to our conclusions by the readiest and most easy method.

This work is divided into eight sections, four of which treat of plane geometry, and four of solid geometry.

The first section, entitled *first principles*, &c. contains the properties of straight lines which meet those of perpendiculars, the theorem upon the sum of the angles of a triangle, the theory of parallel lines, &c.

The second section, entitled *the circle*, treats of the most simple properties of the circle, and those of chords, of tangents, and of the measure of angles by the arcs of a *circle*.

These two sections are followed by the resolution of certain problems relating to the construction of figures.

The third section, entitled the *proportions of figures*, contains the measure of surfaces, their comparison, the properties of a right-angled triangle, those of equiangular triangles, of similar figures, &c. We shall be found fault with perhaps for having blended the properties of lines with those of surfaces; but in this we have followed pretty nearly the example of Euclid, and this order cannot fail of being good, if the propositions are well connected together. This section also is followed by a series of problems relating to the objects of which it treats.

The fourth section treats of *regular polygons and of the measure of the circle.* Two lemmas are employed as the basis of this measure, which is otherwise demonstrated after the manner of Archimedes. We have then given two methods of approximation for squaring the circle, one of which is that of James Gregory. This section is followed by an appendix, in which we have demonstrated that the circle is greater than any rectilineal figure of the same perimeter.

The first section of the second part contains the properties of *planes* and of *solid angles.* This part is very necessary for the understanding of solids and of figures in which different planes are considered. We have endeavoured to render it more clear and more rigorous than it is in common works.

The second section of the second part treats of *polyedrons* and of their measure. This section will be found to be very different from that relating to the same subject in other treatises; we have thought we ought to present it in a manner entirely new.

The third section of this part is an abridged treatise *on the sphere and spherical triangles.* This treatise does not ordinarily make a part of the elements of geometry; still we have thought it proper to consider so much of it as may form an introduction to spherical trigonometry.

The fourth section of the second part treats of *the three round bodies*, which are the sphere, the cone, and the cylinder. The measure of the surfaces and solidities of these bodies is determined by a method analogous to that of Archimedes, and founded, as to surfaces, upon the same principles, which we have endeavoured to demonstrate under the name of *preliminary lemmas.*

At the end of this section is added an appendix to the third section of the second part on *spherical isoperimetrical polygons;* and an appendix to the second and third sections of this part on the *regular polyedrons.*

INTRODUCTION.

In order to abridge the language of geometry particular signs are substituted for the words which most frequently occur; and when we are employed upon any number or magnitude without considering its particular value, but merely with a view to indicate its relation to other magnitudes, or the operations to which it is to be subjected, we distinguish it by a letter of the alphabet, which thus becomes an abridged name for this magnitude.

I. $+$ signifies plus, or added to.

The expression $A+B$ indicates the sum which results from the magnitude represented by the letter A being added to that represented by B, or A plus B.

$-$ signifies minus.

$A-B$ denotes what remains after the magnitude represented by B has been subtracted from that represented by A.

$\times$ signifies multiplied by.

$A\times B$ indicates the product arising from the magnitude represented by A being multiplied by the magnitude represented by B, or A multiplied by B. This product is also sometimes denoted by writing the letters one after the other without any sign, thus AB signifies the same as $A\times B$.

The expression $A\times(B+C-D)$ represents the product of A by the quantity $B+C-D$, the magnitudes included within the parenthesis being considered as one quantity.

$\frac{A}{B}$ indicates the quotient arising from the magnitude represented by A being divided by that represented by B, or A divided by B.

$A=B$ signifies that the magnitude represented by A is equal to that represented by B, or A equal to B.

$A>B$ signifies that the magnitude represented by A exceeds that represented by B, or A greater than B.

$A<B$ signifies A less than B.

$2A$, $3A$, &c., indicate double, triple, &c., of the magnitude represented by A.

II. When a number is multiplied by itself, the result is the *second power*, or *square*, of this number; 5×5, or 25, is the second power, or square, of 5.

The second power therefore is the product of two equal factors; each of these factors is the square root of the product; 5 is the square root of 25.

If the second power be multiplied by its root, the result is the *third power*, or *cube*; 5×25, or 125, is the third power of 5.

The third power is a product formed by the multiplication of three equal factors; each of these factors is the *cube root* of this product; 125 is the product of 5 multiplied twice by itself, or $5 \times 5 \times 5$; and 5 is the cube root of 125.

In general A^2, being an abbreviation of $A \times A$, indicates the second power or square of A.

$\sqrt{A}$ indicates the square root of A, or the number, which being multiplied by itself, produces the number represented by A.

A^3, being an abbreviation of $A \times A \times A$, indicates the third power or cube of A.

$\sqrt[3]{A}$ indicates the cube root of A, or the number which, being multiplied twice by itself, produces the number A.

The square of a line AB is denoted by $\overline{AB}^2$.

The square root of a product $A \times B$ is represented by $\sqrt{A \times B}$.

All numbers are not perfect squares or perfect cubes, that is, they have not square roots or cube roots which can be exactly expressed; 19, for example, as it is between 16, the square of 4, and 25, the square of 5, has for its root a number comprehended between 4 and 5, but which cannot be exactly assigned.

In like manner 89, which is between 64, the cube of 4, and 125, the cube of 5, has for its cube root a number between 4 and 5, but which cannot be exactly assigned. Algebra furnishes methods for approximating, as nearly as we please, the roots of numbers which are not perfect powers.

III. 1. When two proportions have a common ratio, it is evident that the two other ratios may be put into a proportion, since they are each equal to that which is common. If, for example, we have

$$A : B :: C : D,$$
$$E : F :: C : D,$$

then we shall have $A : B :: E : F.$

2. When two proportions have the same antecedents, the consequents may be put into a proportion; for, if we have

$$A : B :: C : D,$$
$$A : E :: C : F,$$

by changing the place of the means, these proportions will become

$$A : C :: B : D,$$
$$A : C :: E : F;$$

whence $B : D :: E : F,$

or $B : E :: D : F.$

IV. Other changes, besides the transposition of terms, may be made among proportionals without destroying the equality of the product of the extremes to that of the means.

1. If to the consequent of a ratio we add the antecedent, and compare this sum with the antecedent, this last will be contained once more than it was in the first consequent; the new ratio then will be equal to the primitive ratio increased by unity. If the same operation be performed upon the two ratios of a proportion, there will evidently result from it two new ratios equal to each other, and consequently a new proportion.

Let there be, for example, the proportion

$$4 : 6 :: 12 : 18$$

we shall have $6 + 4 : 4 :: 18 + 12 : 12,$

or $10 : 4 :: 30 : 12.$

2. If from the consequent of a ratio we subtract the antecedent, and compare the difference with the antecedent, this last will be contained once less than it was in the first consequent; the new ratio will be equal to the primitive ratio diminished by unity. If the same operation be performed upon the two ratios of a proportion, there will result from it two new ratios equal to each other, and consequently a new proportion.

From the proportion

$$4 : 6 :: 12 : 18,$$

we thus deduce $6 - 4 : 4 :: 18 - 12 : 12,$

or $2 : 4 :: 6 : 12.$

There being a proportion among any magnitudes whatever designated by the letters

$$A : B :: C : D,$$

we have, by the above changes,

$$B + A : A :: D + C : C,$$
$$B - A : A :: D - C : C.$$

If we change the place of the means in these results, they will become

$$B + A : D + C :: A : C,$$
$$B - A : D - C :: A : C;$$

but, by the same change, the proportion

$$A : B :: C : D$$

gives also $A : C :: B : D,$

and, since the ratios $A : C$, $B : D$, are equal, we obtain

$$B + A : D + C :: A : C \text{ or } :: B : D,$$
$$B - A : D - C :: A : C \text{ or } :: B : D,$$

a result which may be thus enunciated.

In any proportion whatever, the sum of the two first terms is to the sum of the two last, and the difference of the two first terms is to the difference of the two last, as the first is to the third, or as the second is to the fourth.

Moreover the two ratios $A : C$, $B : D$, being common to the two proportions above obtained, it follows that the other ratios of the same proportions are equal, and that consequently

$$B + A : D + C :: B - A : D - C,$$

or, by changing the place of the means,

$$B + A : B - A :: D + C : D - C;$$

that is, *the sum of the two first terms of a proportion is to their difference, as the sum of the two last is to their difference.*

For example,

$$6 + 4 : 6 - 4 :: 18 + 12 : 18 - 12,$$

or $10 : 2 :: 30 : 6.$

When the proportion

$$A : B :: C : D,$$

is changed into $A : C :: B : D,$

A and B are the antecedents, C and D the consequents; and the proportions

$$B+A:D+C::A:C \text{ or } ::B:D,$$
$$B-A:D-C::A:C \text{ or } ::B:D,$$

answer to the following enunciation;

The sum of the antecedents of a proportion is to the sum of the consequents, and the difference of the antecedents is to the difference of the consequents, as one antecedent is to its consequent;

Whence it follows, that *the sum of the antecedents is to their difference, as the sum of the consequents is to their difference.*

If we have a series of equal ratios

$$A:B::C:D::E:F,$$

by considering only the two first, which form the proportion

$$A:B::C:D,$$

we obtain by what precedes

$$A+C:B+D::A:B;$$

and, since the third ratio $E:F$, is equal to the first $A:B$, we have

$$A+C:B+D::E:F.$$

If we take the sum of the antecedents and that of the consequents in this last proportion, the result will be

$$A+C+E:B+D+F::E:F \text{ or } ::A:B.$$

By proceeding in the same manner with any number of equal ratios, it will be seen, that *the sum of any number whatever of antecedents is to the sum of their consequents, as one antecedent is to its consequent.*

V. Let there be any two proportions

$$A:B::C:D,$$
$$E:F::G:H,$$

if we multiply them in *order*, that is, term by term, the products will form a proportion, thus

$$A\times E:B\times F::C\times G:D\times H,$$

This is evident, since the new ratios $\frac{B\times F}{A\times E}$, $\frac{D\times H}{C\times G}$, are respectively the products of the primitive ratios

$$\frac{B}{A} \text{ and } \frac{F}{E}, \quad \frac{D}{C} \text{ and } \frac{H}{G},$$

which are equal.

If we multiply the proportion

$$A : B :: C : D$$

by

$$A : B :: C . D$$

we shall have (II) $A^2 : B^2 :: C^2 : D^2$,

whence it follows, that *the squares of four proportional quantities form a new proportion.*

By multiplying the proportion

$$A^2 : B^2 :: C^2 : D^2$$

by

$$A : B :: C : D,$$

we shall have

$$A^3 : B^3 :: C^3 : D^3,$$

that is, *the cubes of four proportional quantities form a new proportion.*

VI. When a proportion is said to exist among certain magnitudes, these magnitudes are supposed to be represented, or to be capable of being represented by numbers; if, for example, in the proportion

$$A : B :: C : D,$$

A, B, C, D, denote certain lines, we can always suppose one of these lines, or a fifth, if we please, to answer as a common measure to the whole, and to be taken for unity; then A, B, C, D, will each represent a certain number of units, entire or fractional, commensurable or incommensurable, and the proportion among the lines A, B, C, D, becomes a proportion in numbers.

Hence the product of two lines A and D, which is called also their *rectangle*, is nothing else than the number of linear units contained in A multiplied by the number of linear units contained in B; and we can easily conceive this product to be equal to that which results from the multiplication of the lines B and C.

The magnitudes A and B in the proportion

$$A : B :: C : D,$$

may be of one kind, as lines, and the magnitudes C and D of another kind, as surfaces; still these magnitudes are always to be regarded as numbers; A and B will be expressed in linear units, C and D in superficial units, and the product $A \times D$ will be a number, as also the product $B \times C$.

Indeed, in all the operations, which are made upon proportional quantities, it is necessary to regard the terms of the proportion as so many numbers, each of its proper kind; then we shall have no difficulty in conceiving of these operations and of the consequences which result from them.

ELEMENTS OF GEOMETRY.

Definitions and Preliminary Remarks.

1. GEOMETRY is a science which has for its object the measure of extension.

Extension has three dimensions, length, breadth, and thickness.

2. A *line* is length without breadth.

The extremities of a line are called *points*. A point, therefore, has no extension.

3. A *straight* or *right line* is the shortest way from one point to another.

4. Every line, which is neither a straight line nor composed of straight lines, is a *curved line*.

Thus *AB* (*fig.* 1) is a straight line, *ACDB* is a *broken* line, or one composed of straight lines, and *AEB* is a curved line. Fig. 1.

5. A *surface* is that which has length and breadth, without thickness.

6. A *plane* is a surface, in which any two points being taken, the straight line joining those points lies wholly in that surface.

7. Every surface, which is neither a plane nor composed of planes, is a *curved surface*.

8. A *solid* is that which unites the three dimensions of extension.

9. When two straight lines, *AB*, *AC*, (*fig.* 2), meet, the quantity, whether greater or less, by which they depart from each other as to their position, is called an *angle;* the point of meeting or *intersection* *A*, is the vertex of the angle; the lines *AB*, *AC*, are its *sides*. Fig. 2.

An angle is sometimes denoted simply by the letter at the vertex, as *A;* sometimes by three letters, as *BAC*, or *CAB*, the letter at the vertex always occupying the middle place.

Angles, like other quantities, are susceptible of addition, subtraction, multiplication, and division; thus, the angle *DCE*
Fig. 20. (*fig.* 20) is the sum of the two angles *DCB*, *BCE*, and the angle *DCB* is the difference between the two angles *DCE*, *BCE*.

Fig. 3. 10. When a straight line *AB* (*fig.* 3) meets another straight line *CD* in such a manner that the adjacent angles *BAC*, *BAD*, are equal, each of these angles is called a *right angle*, and the line *AB* is said to be *perpendicular* to *CD*.

Fig. 4. 11. Every angle *BAC* (*fig.* 4), less than a right angle, is an *acute angle*; and every angle, *DEF*, greater than a right angle is an *obtuse angle*.

Fig. 5. 12. Two lines are said to be *parallel* (*fig.* 5), when, being situated in the same plane and produced ever so far both ways, they do not meet.

13. A *plane figure* is a plane terminated on all sides by lines.

If the lines are straight, the space which they contain is
Fig. 6. called a *rectilineal figure*, or *polygon* (*fig.* 6), and the lines taken together make the *perimeter* of the polygon.

14. The polygon of three sides is the most simple of these figures, and is called a *triangle*; that of four sides is called a *quadrilateral*; that of five sides, a *pentagon*; that of six, a *hexagon*, &c.

Fig. 7. 15. A triangle is denominated *equilateral* (*fig.* 7), when the
Fig. 8. three sides are equal, *isosceles* (*fig.* 8), when two only of its sides
Fig. 9. are equal, and *scalene* (*fig.* 9), when no two of its sides are equal.

16. A *right-angled* triangle is that which has one right angle. The side opposite to the right angle is called the *hypothenuse*.
Fig. 10. Thus *ABC* (*fig.* 10) is a triangle right-angled at *A*, and the side *BC* is the hypothenuse.

17. Among quadrilateral figures we distinguish;

Fig. 11. The *square* (*fig.* 11), which has its sides equal and its angles right angles, (See art. 80);

Fig. 12. The *rectangle* (*fig.* 12), which has its angles right angles without having its sides equal (See art. above referred to);

Fig. 13. The *parallelogram* (*fig.* 13), which has its opposite sides parallel;

Fig. 14. The *rhombus* or *lozenge* (*fig.* 14), which has its sides equal without having its angles right angles;

Fig. 15. The *trapezoid* (*fig.* 15), which has two only of its sides parallel.

18. A *diagonal* is a line which joins the vertices of two angles not adjacent, as AC (*fig.* 42). Fig. 42.

19. An *equilateral* polygon is one which has all its sides equal; an *equiangular* polygon is one which has all its angles equal.

20. Two polygons are *equilateral with respect to each other*, when they have their sides equal, each to each, and placed in the same order, that is, when by proceeding round in the same direction the first in the one is equal to the first in the other, the second in the one to the second in the other, and so on. In a similar sense are to be understood two polygons *equiangular with respect to each other*. The equal sides in the first case, and the equal angles in the second, are called *homologous* (A).

21. An *Axiom* is a proposition, the truth of which is self-evident.

A *Theorem* is a truth which becomes evident by a process of reasoning called a *demonstration*.

A *Problem* is a question proposed which requires a solution.

A *Lemma* is a subsidiary truth employed in the demonstration of a theorem, or in the solution of a problem.

The common name of *Proposition* is given indifferently to theorems, problems, and lemmas.

A *Corollary* is a consequence which follows from one or several propositions.

A *Scholium* is a remark upon one or more propositions which have gone before, tending to show their connexion, their restriction, their extension, or the manner of their application.

A *Hypothesis* is a supposition made either in the enunciation of a proposition, or in the course of a demonstration.

Axioms.

22. Two quantities, each of which is equal to a third, are equal to one another.

23. The whole is greater than its part.

24. The whole is equal to the sum of all its parts.

25. Only one straight line can be drawn between two points.

26. Two magnitudes, whether they be lines, surfaces, or solids, are equal, when, being applied the one to the other, they coincide with each other entirely, that is, when they exactly fill the same space.

PART FIRST.

OF PLANE FIGURES.

SECTION FIRST.

First Principles, or the Properties of perpendicular, oblique, and parallel Lines.

THEOREM.

27. All *right angles are equal.*

Demonstration. Let the straight line CD be perpendicular to Fig. 16. AB (*fig.* 16), and GH to EF, the angles ACD, EGH, will be equal.

Take the four distances CA, CB, GE, GF, equal to each other, the distance AB will be equal to the distance EF, and the line EF may be applied to AB, so that the point E will fall upon A, and the point F upon B. These two lines, thus placed, will coincide with each other throughout; otherwise there would be two straight lines between A and B, which is impossible (25). The point G therefore, the middle of EF, will fall upon the point C, the middle of AB. The side GE being thus applied to CA, the side GH will fall upon CD; for, let us suppose, if it be possible, that it falls upon a line CK, different from CD; since, by hypothesis (10), the angle $EGH = HGF$,

it follows that $ACK = KCB$.

But $ACK > ACD$,

and $KCB < BCD$;

besides, by hypothesis,

$ACD = BCD$;

hence $ACK > KCB$.

and the line GH cannot fall upon a line CK different from CD; consequently it falls upon CD, and the angle EGH upon ACD, and EGH is equal to ACD; therefore all right angles are equal.

THEOREM.

Fig. 17. 28. *A straight line* CD (fig. 17), *which meets another straight line* AB, *makes with it two adjacent angles* ACD, BCD, *which, taken together, are equal to two right angles.*

Demonstration. At the point C, let CE be perpendicular to AB. The angle ACD is the sum of the angles ACE, ECD; therefore $ACD + BCD$ is the sum of the three angles ACE, ECD, BCD. The first of these is a right angle, and the two others are together equal to a right angle; therefore the sum of the two angles ACD, BCD, is equal to two right angles.

29. *Corollary* I. If one of the angles ACD, BCD, is a right angle, the other is also a right angle.

30. *Corollary* II. If the line DE (*fig.* 18) is perpendicular to AB; reciprocally, AB is also perpendicular to DE. Fig. 18.

For, since DE is perpendicular to AB, it follows that the angle ACD is equal to its adjacent angle DCB, and that they are both right angles. But, since the angle ACD is a right angle, it follows that its adjacent angle ACE is also a right angle; therefore the angle $ACE = ACD$, and AB is perpendicular to DE.

31. *Corollary* III. All the successive angles, BAC, CAD, DAE, EAF, (*fig.* 34), formed on the same side of the straight line BF, are together equal to two right angles; for their sum is equal to that of the two angles BAM, MAF; AM being perpendicular to BF. Fig. 34.

THEOREM.

32. *Two straight lines, which have two points common, coincide throughout, and form one and the same straight line.* 3

Demonstration. Let the two points, which are common to the two lines, be A and B (*fig.* 19). In the first place it is evident that they must coincide entirely between A and B; otherwise, two straight lines could be drawn from A to B, which is impossible (25). Now let us suppose, if it be possible, that the lines, when produced, separate from each other at a point C, the one becoming CD, and the other CE. At the point C, let CF be drawn, so as to make the angle ACF, a right angle; then, ACD being a straight line, the angle FCD is a right angle (29); and, because ACE is a straight line, the angle FCE is a right angle. But the part FCE cannot be equal to the whole FCD; whence straight lines, which have two points common A and B, cannot separate the one from the other, when produced; therefore they must form one and the same straight line. Fig. 19.

THEOREM.

4

Fig. 20. 33. *If two adjacent angles* ACD, DCB (fig. 20), *are together equal to two right angles, the two exterior sides* AC, CB, *are in the same straight line.*

Demonstration. For if *CB* is not the line *AC* produced, let *CE* be that line produced; then, *ACE* being a straight line, the angles *ACD*, *DCE*, are together equal to two right angles (28); but, by hypothesis, the angles *ACD*, *DCB*, are together equal to two right angles, therefore $ACD + DCB = ACD + DCE$. Take away the common angle *ACD*, and there will remain the part *DCB* equal to the whole *DCE*, which is impossible; therefore *CB* is the line *AC* produced.

THEOREM.

5

Fig. 21. 34. *Whenever two straight lines* AB, DE (fig. 21), *cut each other, the angles opposite*[†] *to each other at the vertex are equal.*

Demonstration. Since *DE* is a straight line, the sum of the angles *ACD*, *ACE*, is equal to two right angles; and, since *AB* is a straight line, the sum of the angles *ACE*, *BCE*, is equal to two right angles; therefore $ACD + ACE = ACE + BCE$; from each of these take away the common angle *ACE*, and there will remain the angle *ACD* equal to its opposite angle *BCE*.

It may be demonstrated, in like manner, that the angle *ACE* is equal to its opposite angle *BCD*.

35. *Scholium.* The four angles, formed about a point by two straight lines which cut each other, are together equal to four right angles; for the angles *ACE*, *BCE*, taken together, are equal to two right angles; also the other angles *ACD*, *BCD*, are together equal to two right angles.

Fig. 22. In general, if any number of straight lines, as *CA*, *CB* (*fig.* 22), &c., meet in the same point *C*, the sum of all the successive angles, *ACB*, *BCD*, *DCE*, *ECF*, *FCA*, will be equal to four right angles. For, if at the point *C*, four right angles be formed by two lines perpendicular to each other, they will comprehend the same space as the successive angles, *ACB*, *BCD*, &c.

[†] These are often called *vertical* angles.

THEOREM.

6

36. *Two triangles are equal, when two sides and the included angle of the one are equal to two sides and the included angle of the other, each to each.*

Demonstration. In the two triangles ABC, DEF (*fig.* 23), let the angle A be equal to the angle D, the side AB equal to the side DE, and the side AC equal to the side DF; the two triangles ABC, DEF, will be equal. Fig. 23.

Indeed the triangles may be so placed, the one upon the other, that they shall coincide throughout. If, in the first place, we apply the side DE to its equal AB, the point D will fall upon A, and the point E upon B. But, since the angle D is equal to the angle A, when the side DE is placed apon AB, the side DF will take the direction AC; moreover DF is equal to AC; therefore the point F will fall upon C, and the third side EF will exactly coincide with the third side BC; therefore the triangle DEF is equal to the triangle ABC (26).

37. *Corollary.* When, in two triangles, these three things are equal, namely, the angle $A = D$, the side $AB = DE$, and the side $AC = DF$, we may thence infer, that the other three are also equal, namely, the angle $B = E$, the angle $C = F$, and the side $BC = EF$.

THEOREM.

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38. *Two triangles are equal, when a side and the two adjacent angles of the one, are equal to a side and the two adjacent angles of the other, each to each.*

Demonstration. Let the side BC (*fig.* 23) be equal to the side EF, the angle B equal to the angle E, and the angle C equal to the angle F; the triangle ABC will be equal to the triangle DEF. Fig. 23.

For, in order to apply the one to the other, let EF be placed upon its equal BC, the point E will fall upon B and the point F upon C. Then because the angle E is equal to the angle B, the side ED will take the direction BA, and therefore the point D will be somewhere in BA; also because the angle F is equal to C, the side FD will take the direction CA, and therefore the point D will be somewhere in CA; whence the point D, which must be at the same time in the lines BA and CA, can only be at their intersection A; therefore the two triangles ABC,

DEF, coincide, the one with the other, and are equal in all respects.

39. *Corollary.* When, in two triangles, these three things are equal, namely, $BC = EF$, $B = E$, and $C = F$, we may thence infer that the other three are also equal, namely, $AB = DE$, $AC = DF$, and $A = D$.

THEOREM.

8 40. *One side of a triangle is less than the sum of the other two.*

Fig. 23. *Demonstration.* The straight line BC (*fig.* 23), for example, is the shortest way from B to C (3); BC therefore is less than $BA + AC$.

THEOREM.

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Fig. 24. 41. *If from a point* O (fig. 24), *within a triangle* ABC, *there be drawn straight lines* OB, OC, *to the extremities of* BC, *one of its sides, the sum of these lines will be less than that of* AB, AC, *the two other sides.*

Demonstration. Let BO be produced till it meet the side AC in D; the straight line OC is less than $OD + DC$; to each of these add BO, and $BO + OC < BO + OD + DC$; that is

$$BO + OC < BD + DC.$$

Again, $BD < BA + AD$; to each of these add DC, and we shall have $BD + DC < BA + AC$. But it has just been shown that $BO + OC < BD + DC$, much more then is

$$BO + OC < BA + AC.$$

THEOREM.

10

Fig. 25. 42. *If two sides* AB, AC (fig. 25), *of a triangle* ABC, *are equal to two sides* DE, DF, *of another triangle* DEF, *each to each; if, at the same time, the angle* BAC, *contained by the former, is greater than the angle* EDF, *contained by the latter; the third side* BC *of the first triangle, will be greater than the third side* EF *of the second.*

Demonstration. Make the angle $CAG = D$, take $AG = DE$, and join CG, then the triangle GAC is equal to the triangle EDF (36), and therefore $CG = EF$. Now there may be three cases, according as the point G falls without the triangle ABC, on the side BC, or within the triangle.

Case I. Because $GC < GI + IC$, and $AB < AI + IB$, therefore $GC + AB < GI + AI + IC + IB$, that is,

$$GC + AB < AG + BC.$$

From one of these take away AB, and from the other its equal AG, and there remains $GC < BC$; therefore $EF < BC$.

Case II. If the point G (*fig.* 26) fall upon the side BC, then it is evident that GC, or its equal EF, is less than BC. **Fig. 26.**

Case III. If the point G (*fig.* 27) fall within the triangle BAC, then $AG + GC < AB + BC$ (41), therefore, taking away the equal quantities, AG, AB, we shall have $GC < BC$, or $EF < BC$. **Fig. 27.**

THEOREM.

43. *Two triangles are equal, when the three sides of the one are equal to the three sides of the other, each to each.* 11

Demonstration. Let the side $AB = DE$ (*fig.* 23,) $AC = DF$, $BC = EF$; then the angles will be equal, namely, $A = D$, $B = E$, and $C = F$. **Fig. 23.**

For, if the angle A were greater than the angle D, as the sides AB, AC, are equal to the sides DE, DF, each to each, the side BC would be greater than EF (42); and if the angle A were less than the angle D, then the side BC would be less than EF; but BC is equal to EF, therefore the angle A can neither be greater nor less than the angle D, that is, it is equal to it. In the same manner it may be proved, that the angle $B = E$, and that the angle $C = F$.

44. *Scholium.* It may be remarked, that equal angles are opposite to equal sides; thus, the equal angles A and D are opposite to the equal sides BC and EF.

THEOREM.

45. *In an isosceles triangle the angles opposite to the equal sides are equal.* 12

Demonstration. Let the side $AB = AC$ (*fig.* 28), then will the angle C be equal to B. **Fig. 28.**

Draw the straight line AD from the *vertex* A to the point D the middle of the *base* BC; the two triangles ABD, ADC, will have the three sides of the one, equal to the three sides of the other, each to each, namely, AD common to both, $AB = AC$, by hypothesis, and $BD = DC$, by construction; therefore (43) the angle B is equal to the angle C.

46. *Corollary.* An equilateral triangle is also equiangular, that is, it has its angles equal.

47. *Scholium.* From the equality of the triangles ABD, ACD, it follows, that the angle $BAD = DAC$, and that the angle $BDA = ADC$; therefore these two last are right angles. *Hence a straight line drawn from the vertex of an isosceles triangle, to the middle of the base, is perpendicular to that base, and divides the vertical angle into two equal parts.*

In a triangle that is not isosceles, any one of its sides may be taken indifferently for a base; and then its vertex is that of the opposite angle. In an isosceles triangle, the base is that side which is not equal to one of the others.

THEOREM.

13 48. *Reciprocally, if two angles of a triangle are equal, the opposite sides are equal, and the triangle is isosceles.*

Fig. 29. *Demonstration.* Let the angle $ABC = ACB$ (*fig.* 29), the side AC will be equal to the side AB.

For, if these sides are not equal, let AB be the greater. Take $BD = AC$, and join DC. The angle DBC is, by hypothesis, equal to ACB, and the two sides DB, BC, are equal to the two sides AC, CB, each to each; therefore the triangle DBC is equal to the triangle ACB (36); but a part cannot be equal to the whole; therefore the sides AB, AC, cannot be unequal; that is, they are equal, and the triangle is isosceles.

THEOREM.

14 49. *Of the two sides of a triangle, that is the greater, which is opposite to the greater angle; and conversely, of the two angles of a triangle, that is the greater, which is opposite to the greater side.*

Fig. 30. *Demonstration.* 1. Let the angle $C > B$ (*fig.* 30), then will the side AB, opposite to the angle C, be greater than the side AC, opposite to the angle B.

Draw CD making the angle $BCD = B$. In the triangle BDC, BD is equal to DC (48); but $AD + DC > AC$, and

$$AD + DC = AD + DB = AB, \text{ therefore } AB > AC.$$

2. Let the side $AB > AC$, then will the angle C, opposite to the side AB, be greater than the angle B, opposite to the side AC. For, if C were less than B, then according to what has just been demonstrated we should have $AB < AC$, which is contrary to the hypothesis; and if C were equal to B, then it would

follow, that $AC = AB$ (48), which is also contrary to the hypothesis; whence the angle C can be neither less than B, nor equal to it; it is therefore greater.

THEOREM. 15

50. *From a given point* A (fig. 31), *without a straight line* DE, *only one perpendicular can be drawn to that line.* Fig. 31.

Demonstration. If it be possible, let there be two AB and AC; produce one of them AB, so that $BF = AB$, and join CF.

The triangle CBF is equal to the triangle ABC. For the angle CBF is a right angle (29), as well as CBA, and the side $BF = BA$; therefore the triangles are equal (36), and the angle $BCF = BCA$. But BCA is, by hypothesis, a right angle; therefore BCF is also a right angle. But, if the adjacent angles BCA, BCF, are together equal to two right angles, ACF must be a straight line (33); and hence it would follow that two straight lines ACF, ABF, might be drawn between the same two points A and F, which is impossible (25); it is then equally impossible to draw two perpendiculars from the same point to the same straight line.

51. *Scholium.* Through the same point C (*fig.* 17), in the line AB, it is also impossible to draw two perpendiculars to that line; for, if CD and CE were these two perpendiculars, the angle DCB would be a right angle as well as BCE; and a part would be equal to the whole. Fig. 17.

THEOREM. 16

52. *If from a point* A (fig. 31), *without a straight line* DE, *a perpendicular* AB *be drawn to that line, and also different oblique lines* AE, AC, AD, &c., *to different points of the same line;* Fig. 31.

1. *The perpendicular* AB *is less than any one of the oblique lines;*

2. *The two oblique lines* AC, AE, *which meet the line* DE *on opposite sides of the perpendicular, and at equal distances* BC, BE, *from it, are equal to one another;*

3. *Of any two oblique lines* AC, AD, *or* AE, AD, *that which is more remote from the perpendicular is the greater.*

Demonstration. Produce the perpendicular AB, so that $BF = BA$, and join FC, FD.

1. The triangle BCF is equal to the triangle BCA; for the right angle $CBF = CBA$, the side CB is common, and the side $BF = BA$; therefore the third side CF is equal to the third side AC (36). But $AF < AC + CF$ (40), and AB half of AF is less than AC half of $AC + CF$, that is, the perpendicular is less than any one of the oblique lines.

2. If $BE = BC$, then, as AB is common to the two triangles ABE, ABC, and the right angle $ABE = ABC$, the triangle ABE is equal to the triangle ABC, and $AE = AC$.

3. In the triangle DFA, the sum of the sides AD, DF, is greater than the sum of the sides AC, CF (41); therefore AD half of $AD + DF$ is greater than AC half of $AC + CF$, and the oblique line, which is more remote from the perpendicular, is greater than that which is nearer.

53. *Corollary* I. The perpendicular measures the distance of any point from a straight line.

54. *Corollary* II. From the same point, there cannot be drawn three equal straight lines terminating in a given straight line; for, if this could be done, there would be on the same side of the perpendicular two equal oblique lines, which is impossible.

THEOREM.

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Fig. 32. 55. *If from the point* C (fig. 32), *the middle of the straight line* AB, *a perpendicular* EF *be drawn*; 1. *each point in the perpendicular* EF *is equally distant from the two extremities of the line* AB; 2. *any point without the perpendicular is at unequal distances from the same extremities* A *and* B.

Demonstration. 1. Since $AC = CB$, the two oblique lines AD, DB, are drawn to points which are at the same distance from the perpendicular. They are therefore equal (52). The same reasoning will apply to the two oblique lines AE, EB, also to AF, FB, &c. Whence each point in the perpendicular EF is equally distant from the extremities of the line AB.

2. Let I be a point out of the perpendicular; join IA IB, one of these lines must cut the perpendicular in D; join DB, then $DB = DA$. But the line $IB < ID + DB$ and

$$ID + DB = ID + DA = IA;$$

therefore $IB < IA$; that is, any point without the perpendicular is at unequal distances from the extremities of AB.

THEOREM.

8

56. *Two right-angled triangles are equal, when the hypothenuse and a side of the one are equal to the hypothenuse and a side of the other, each to each.*

Demonstration. Let the hypothenuse $AC = DF$ (*fig.* 33), and Fig. 33. the side $AB = DE$; the right-angled triangle ABC will be equal to the right-angled triangle DEF.

The proposition will evidently be true, if the third side BC be equal to the third side EF. If it be possible, let these sides be unequal, and let BC be the greater. Take $BG = EF$, and join AG; then the triangle ABG is equal to the triangle DEF, for the right angle B is equal to the right angle E, the side $AB = DE$ and the side $BG = EF$; therefore these two triangles being equal (36), $AG = DF$; and, by hypothesis, $DF = AC$; whence $AG = AC$. But AG cannot be equal to AC (52); therefore it is impossible that BC should be unequal to EF, that is, it is equal to it, and the triangle ABC is equal to the triangle DEF.

THEOREM.

9

57. *If two straight lines,* AC, BD, (fig. 35), *are perpendicular to* Fig. 35. *a third* AB, *these two lines are parallel, that is, they will not meet, however far they are produced* (12).

Demonstration. If they could meet in a point O on one side or the other of the line AB, there would be two perpendiculars drawn from the same point O to the same straight line AB, which is impossible (50).

LEMMA.

10

58. *The straight line* BD (fig. 35), *being perpendicular to* AB, Fig. 35. *if another straight line* AE *make with* AB *an acute angle* BAE, *the straight lines* BD, AE, *being produced sufficiently far, will meet.*

Demonstration. From any point F, taken in the direction AE, let fall upon AB the perpendicular FG; the point G will not fall upon A, since the angle FAB is less than a right angle; still less can it fall upon H in BA produced, for then there would be two perpendiculars KA, KH, let fall from the same point K upon the same straight line AH. The point G then must fall, as the figure represents it, in the direction AB.

Let there be taken in the line AE another point L at a distance AL greater than AF, and let there be drawn to AB the perpendicular LM; it may be shown, as in the preceding case, that the point M can neither fall upon G nor upon any point in the direction GA, so that the distance AM will be necessarily greater than AG.

It may be observed, moreover, that if the figure is constructed with care, and AL be taken double of AF, we shall find that AM is exactly double of AG; also, if AL be taken triple of AF, we shall find that AG is triple of AG, and in general there will always be the same ratio between AM and AG, that there is between AL and AF. From this proportion it follows, not only that the right line AE, must meet BD, if the two lines are produced sufficiently far, but also that we may even assign upon AE the distance of the point of meeting of the two lines. This distance will be the fourth term of the proportion,

$$AG : AB :: AF : x.$$

59. *Scholium.* The foregoing explanation, founded upon a relation which is not deduced from reasoning merely, and for which recourse is had to measures taken upon a figure accurately constructed, has not the same degree of strictness, as the other demonstrations of elementary geometry. It is given here only as a simple method, by which one may satisfy himself of the truth of the proposition. We shall resume the subject with a view to a rigorous demonstration in the third of the notes subjoined to these elements.

THEOREM.

Fig. 36. 60. *If two straight lines* AC, BD, (fig. 36), *make with a third* AB *two interior angles* CAB, ABD, *the sum of which is equal to two right angles, the two lines* AC, BD, *are parallel.*

Demonstration. From the point G, the middle of AB, draw the straight line EGF perpendicular to AC; this line will be perpendicular to BD. Indeed the sum $GAE + GBD$ is, by hypothesis, equal to two right angles, and the sum $GBF + GBD$ is also equal to two right angles (28); taking therefore from each GBD we shall have the angle $GAE = GBF$. Besides, the angles AGE, BGF, are equal, being vertical angles; therefore the triangles AGE, BGF, have a side and the two adjacent angles of the one respectively equal to a side and the two adjacent angles of the other; they are therefore equal (38), and the angle

$$BFG = AEG;$$

but the angle AEG is, by construction, a right angle, consequently the straight lines AC, BD, are perpendicular to the same straight line EF, therefore they are parallel (57).

THEOREM.

61. *If two straight lines*, AI, BD (fig. 36), *make with a third line* AB *two interior angles* BAI, ABD, *the sum of which is less than two right angles, the lines* AI, BD, *produced, will meet.* Fig. 36.

Demonstration. Draw AC making the angle $CAB = ABF$, that is, in such a manner as to make the two angles CAB, ABD, together equal to two right angles, and finish the construction, as in the preceding theorem. Since the angle AEK is a right angle, AE is a perpendicular, and consequently less than the oblique line, AK; whence in the triangle AEK the angle AKE, opposite to the side AE, is less than the right angle AEK, opposite to the side AK (49). Therefore the angle IKF, equal to the angle AKE, is less than a right angle, and the lines KI, FD, being produced, must meet (58).

62. *Scholium.* If the lines AM and BD make with AB two angles BAM, ABD, the sum of which is greater than two right angles, then the two lines AM, BD, would not meet above AB, but they would meet below it. For the two angles BAM, BAN, would together be equal to two right angles, as also the two angles, ABD, ABF, and the four angles taken together would be equal to four right angles. But the sum of the two angles BAM, ABD, is greater than two right angles, therefore the sum of the two remaining ones BAN, ABF, is less than two right angles, and the two right lines AN, BF, being produced, must meet.

63. *Corollary.* Through a given point A only one line can be drawn parallel to a given line BD. For there can be only one line AC, which makes the sum of the two angles BAC, ABD, equal to two right angles; this is the parallel required, every other line AI or AM would make the sum of the interior angles less or greater than two right angles, therefore it would meet the line BD.

THEOREM.

Fig. 37. 64. *If two parallel straight lines* AB, CD (fig. 37), *meet a third line* EF, *the sum of the interior angles upon the same side* AGH, GHC, *will be equal to two right angles.*

Demonstration. If this sum were greater or less than two right angles, the two straight lines *AB*, *CD*, would meet on one side or the other of *EF*, and would not be parallel (61).

65. *Corollary* I. If *GHC* be a right angle, *AGH* will also be a right angle; therefore every line, which is perpendicular to one of the parallels, is also perpendicular to the other.

66. *Corollary* II. Since the sum $AGH + GHC$ is equal to two right angles, and the sum $GHD + GHC$ is also equal to two right angles, if we take away the common part *GHC*, we shall have the angle $AGH = GHD$. Besides, $AGH = BGE$, and

$$GHD = CHF \ (34);$$

therefore the four acute angles *AGH*, *BGE*, *GHD*, *CHF*, are equal to each other; the same may be proved with respect to the four obtuse angles *AGE*, *BGH*, *GHC*, *DHF*. It may be observed, moreover, that by adding one of the four acute angles to one of the four obtuse angles, the sum will always be equal to two right angles.

67. *Scholium.* The angles of which we have been speaking, compared, two and two, take different names. We have already called the angles *AGH*, *GHC*, *interior upon the same side;* the angles *BGH*, *GHD*, have the same name; the angles *AGH*, *GHD*, are called *alternate-internal*, or simply *alternate;* the same may be said of the angles *BGH*, *GHC*. Lastly, we denominate *internal-external* the angles *EGB*, *GHD*, and *EGA*, *GHC*, and *alternate-external* *EGB*, *CHF*, and *AGE*, *DHF*. This being premised, we may regard the following propositions as already demonstrated.

1. The two interior angles upon the same side, taken together, are equal to two right angles.

2. The alternate-internal angles are equal, as also the internal-external, and the alternate-external.

Reciprocally, if in this second case, two angles of the same name are equal, we may infer that the lines to which they are referred are parallel. Let there be, for example, the angle $AGH = GHD$; since $GHC + GHD$ is equal to two right angles, we have also *AGH* + *GHC* equal to two right angles, therefore the lines *AG*, *CH*, are parallel (60).

THEOREM.

68. *Two lines* AB, CD (fig. 38), *which are parallel to a third* EF, *are parallel to one another.* Fig. 38.

Demonstration. Draw *PQR* perpendicular to *EF*. Then, since *AB* is parallel to *EF*, the line *PR* will be perpendicular to *AB* (65); also, since *CD* is parallel to *EF*, the line *PR* will be perpendicular to *CD*. Consequently *AB* and *CD* are perpendicular to the same straight line *PQ*, therefore they are parallel (57).

THEOREM.

69. *Two parallel lines are throughout at the same distance from each other.*

Demonstration. The two parallels *AB*, *CD* (*fig.* 39), being given, if through two points taken at pleasure we erect upon *AB* the two perpendiculars *EG*, *FH*, the straight lines *EG*, *FH*, will be at the same time perpendicular to *CD* (65); moreover these straight lines will be equal to each other. Fig. 39.

For, by drawing *GF*, the angles *GFE*, *FGH*, considered with reference to the parallels *AB*, *CD*, being alternate-internal angles (67), are equal; also since the straight lines *EG*, *FH*, are perpendicular to the same straight line *AB* and consequently parallel to each other, the angles *EGF*, *GFH*, considered with reference to the parallels *EG*, *FH*, being alternate-internal angles, are equal. The two triangles then *EFG*, *FGH*, have a side and the two adjacent angles of the one equal to a side and the two adjacent angles of the other, each to each; these two triangles are therefore equal (38); and the side *EG*, which measures the distance of the parallels *AB*, *CD*, at the point *E*, is equal to the side *FH*, which measures the distauce of the same parallels at the point *F*.

THEOREM.

70. *If two angles*, BAC, DEF (fig. 40), *have their sides parallel, each to each, and directed the same way, these two angles will be equal.* Fig. 40.

Demonstration. Produce DE, if it be necessary, till it meet AC in G; the angle DEF is equal to DGC, because EF is parallel to GC (67); the angle DGC is equal to BAC, because DG is parallel to AB; therefore the angle DEF is equal to BAC.

71. *Scholium.* There is a restriction in this proposition, namely, that the side EF be directed the same way as AC, and ED the same way as AB; the reason is this; if we produce FE toward H, the angle DEH would have its sides parallel to those of the angle BAC, but the two angles would not be equal. In this case the angle DEH and the angle BAC would together make two right angles.

THEOREM.

72. *In every triangle the sum of the three angles is equal to two right angles.*

Fig. 41. *Demonstration.* Let ABC (*fig.* 41) be any triangle; produce the side CA toward D, and draw to the point A the straight line AE parallel to BC.

Because AE, CB, are parallel, the angles ACB, DAE, considered with reference to the line CAD, are equal, being internal-external angles (67); in like manner ABC, BAE, considered with reference to the line AB, are equal, being alternate-internal angles; consequently the three angles of the triangle ABC make the same sum as the three angles CAB, BAE, EAD; therefore this sum is equal to two right augles (31).

73. *Corollary* I. Two angles of a triangle being given, or only their sum, the third will be known by subtracting the sum of these angles from two right angles.

74. *Corollary* II. If two angles of one triangle are equal to two angles of another triangle, each to each, the third of the one will be equal to the third of the other, and the two triangles will be equiangular.

75. *Corollary* III. In a triangle there can be only one right angle; for if there were two, the third angle must be nothing; still less then can a triangle have more than one obtuse angle.

76. *Corollary* IV. In every right-angled triangle the sum of the acute angles is equal to a right angle.

77. *Corollary* V. Every equilateral triangle, as it must be also equiangular (45), has each of its angles equal to a third of two

right angles, so that if a right angle be expressed by unity, the angle of an equilateral triangle will be expressed by $\frac{2}{3}$.

78. *Corollary* vi. In every triangle *ABC* (*fig.* 41) the exterior Fig. 41. angle *BAD* is equal to the two opposite interior angles *B* and *C*; for, *AE* being parallel to *BC*, the part *BAE* is equal to the angle *B*, and the other part *DAE* is equal to the angle *C* (67).

THEOREM.

79. *The sum of all the interior angles of a polygon is equal to as many times two right angles as there are units in the number of sides minus two.*

Demonstration. Let *ABCDE* &c. (*fig.* 42) be the proposed Fig. 42. polygon; if from the vertex of the angle *A* we draw to the vertices of the opposite angles the diagonals *AC*, *AD*, *AE*, &c., it is evident, that the polygon will be divided into five triangles, if it have seven sides, and into six, if it have eight, and in general into as many triangles wanting two, as the polygon has sides; for these triangles may be considered as having for their common vertex the point *A*, and for their bases the different sides of the polygon, except the two which form the angle *BAG*. We see, at the same time, that the sum of the angles of all these triangles does not differ from the sum of the angles of the polygon; therefore this last sum is equal to as many times two right angles, as there are triangles, that is, as there are units in the number of sides of the polygon minus two.

80. *Corollary* i. The sum of the angles of a quadrilateral is equal to two right angles multiplied by 4 — 2, which makes four right angles; therefore, if all the angles of a quadrilateral are equal, each of them will be a right angle, which justifies the definition of a square and rectangle (17).

81. *Corollary* ii. The sum of the angles of a pentagon is equal to two right angles multiplied by 5 — 2, which makes 6 right angles; therefore, when a pentagon is equiangular, each angle is equal to a fifth of six right angles, or $\frac{6}{5}$ of one right angle.

82. *Corollary* iii. The sum of the angles of a hexagon is equal to $2 \times (6 - 2)$, or 8, right angles; therefore, in an equiangular hexagon, each angle is the sixth of eight right angles, or $\frac{4}{3}$ of one right angle. The process may be easily extended to other polygons.

83. *Scholium.* If we would apply this proposition to polygons, which have any *re-entering*† angles, each of these angles is to be considered as greater than two right angles. But, in order to avoid confusion we shall confine ourselves in future to those polygons, which have only *saliant* angles, and which may be called *convex polygons.* Every convex polygon is such, that a straight line, however drawn, cannot meet the perimeter in more than two points.

THEOREM.

84. *The opposite sides of a parallelogram are equal, and the opposite angles also are equal.*

Fig. 44. *Demonstration.* Draw the diagonal *BD* (*fig.* 44); the two triangles *ADB*, *DBC*, have the side *BD* common; moreover, on account of the parallels *AD*, *BC*, the angle $ADB = DBC$ (67), and on account of the parallels *AB*, *CD*, the angle $ABD = BDC$; therefore the two triangles *ADB*, *DBC*, are equal (38); consequently the side *AB* opposite to *ADB* is equal to the side *DC* opposite to the equal angle *DBC*, and likewise the third side *AD* is equal to the third side *BC*; therefore the opposite sides of a parallelogram are equal.

Again, from the equality of the same triangles it follows, that the angle $A = C$, and also that the angle *ADC*, composed of the two angles *ADB*, *BDC*, is equal to the angle *ABC*, composed of the two angles *DBC*, *ABD*; therefore the opposite angles of a parallelogram are equal.

85. *Corollary.* Hence two parallels *AB*, *CD*, comprehended between two other parallels *AD*, *BC*, are equal.

THEOREM.

Fig. 44. 86. *If, in a quadrilateral* ABCD (fig. 44), *the opposite sides are equal, namely,* AB = CD, *and* AD = CB, *the equal sides will be parallel, and the figure will be a parallelogram.*

Demonstration. Draw the diagonal *BD*; the two triangles *ABD*, *BDC*, have the three sides of the one equal to the three sides of the other, each to each, they are therefore equal, and the angle *ADB* opposite to the side *AB* is equal to the angle *DBC* opposite to the side *CD*; consequently the side *AD* is parallel to *BC* (67). For a similar reason *AB* is parallel to *CD*; therefore the quadrilateral *ABCD* is a parallelogram.

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† A re-entering angle is one whose vertex is directed inward, as Fig. 43. *CDE* (*fig.* 43), while a saliant angle has its vertex directed outward as *ABC*.

THEOREM.

87. *If two opposite sides* AB, CD (fig. 44), *of a quadrilateral are equal and parallel, the two other sides will also be equal and parallel, and the figure* ABCD *will be a parallelogram.* Fig. 44.

Demonstration. Let the diagonal *BD* be drawn; since *AB* is parallel to *CD*, the alternate angles *ABD*, *BDC*, are equal (67). Besides, the side $AB = CD$, and the side *DB* is common, therefore the triangle *ABD* is equal to the triangle *DBC* (36), and the side $AD = BC$, the angle $ADB = DBC$, and consequently *AD* is parallel to *BC*; therefore the figure *ABCD* is a parallelogram.

THEOREM.

88. *The two diagonals* AC, DB (fig. 45), *of a parallelogram mutually bisect each other.* Fig. 45.

Demonstration. If we compare the triangle *ADO* with the triangle *COB*, we find the side $AD = CB$, and the angle

$$ADO = CBO \ (67);$$

also the angle $DAO = OCB$; therefore these two triangles are equal (38), and consequently *AO*, the side opposite to the angle *ADO*, is equal to *OC*, the side opposite to the angle *OBC*; *DO* likewise is equal to *OB*.

89. *Scholium.* In the case of the rhombus, the sides *AB*, *BC*, being equal, the triangles *AOB*, *OBC*, have the three sides of the one equal to the three sides of the other, each to each, and are consequently equal; whence it follows, that the angle

$$AOB = BOC,$$

and that thus the two diagonals of a rhombus cut each other mutually at right angles.

SECTION SECOND.

Of the Circle and the Measure of Angles.

DEFINITIONS.

90. THE *circumference of a circle* is a curved line all the points of which are equally distant from a point within called the *centre*.

The circle is the space terminated by this curved line*.

Fig. 46. 91. Every straight line *CA*, *CE*, *CD* (*fig.* 46), &c. drawn from the centre to the circumference is called a *radius* or *semidiameter*, and every straight line, as *AB*, which passes through the centre and is terminated each way by the circumference, is called a *diameter*.

By the definition of a circle the radii are all equal, and all the diameters also are equal and double of the radius.

92. An *arc* of a circle is any portion of its circumference, as *FHG*.

The *chord* or *subtense* of an arc is the straight line *FG*, which joins its extremities**.

93. A *segment* of a circle is the portion comprehended between an arc and its chord.

94. A *sector* is the part of a circle comprehended between an arc *DE* and the two radii *CD*, *CE*, drawn to the extremities of this arc.

95. A *straight line* is said to be *inscribed in a circle*, when its extremities are in the circumference of the circle, as *AB* (*fig.* 47). Fig. 47.

An *inscribed angle* is one whose vertex is in the circumference, and which is formed by two chords, as *BAC*.

An *inscribed triangle* is a triangle whose three angles have their vertices in the circumference of the circle, as *BAC*.

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* In common discourse the circle is sometimes confounded with its circumference; but it will always be easy to preserve the exactness of these expressions by recollecting that the circle is a surface which has length and breadth, while the circumference is only a line.

** The same chord, as *FG*, corresponds to two arcs, and consequently to two segments; but, in speaking of these, the smaller is always to be understood, when the contrary is not expressed.

And in general an *inscribed figure* is one, all whose angles have their vertices in the circumference of the circle. In this case, the circle is said to be *circumscribed* about the figure.

96. A *secant* is a line, which meets the circumference in two points, as *AB* (*fig.* 48). Fig. 48.

97. A *tangent* is a line which has only one point in common with the circumference, as *CD*.

The common point *M* is called *the point of contact*.

Also two circumferences are *tangents* to each other (*fig.* 59, 60), when they have only one point common. Fig. 59, 60.

A polygon is said to be *circumscribed* about a circle, when all its sides are tangents to the circumference; and in this case the circle is said to be *inscribed* in the polygon.

THEOREM.

98. *Every diameter* AB (fig. 49) *bisects the circle and its circumference.* Fig. 49.

Demonstration. If the figure *AEB* be applied to *AFB*, so that the base *AB* may be common to both, the curved line *AEB* must fall exactly upon the curved line *AFB*; otherwise, there would be points in the one or the other unequally distant from the centre, which is contrary to the definition of a circle.

THEOREM.

99. *Every chord is less than the diameter.*

Demonstration. If the radii *CA*, *CD* (*fig.* 49), be drawn from the centre to the extremities of the chord *AD*, we shall have the straight line $AD < AC + CD$, that is, $AD < AB$ (91). Fig. 49.

100. *Corollary.* Hence the greatest straight line that can be inscribed in a circle is equal to its diameter.

THEOREM.

101. *A straight line cannot meet the circumference of a circle in more than two points.*

Demonstration. If it could meet it in three, these three points being equally distant from the centre, there might be three equal straight lines drawn from a given point to the same straight line, which is impossible (54).

THEOREM.

102. *In the same circle, or in equal circles, equal arcs are subtended by equal chords, and conversely, equal chords subtend equal arcs.*

Fig. 50. *Demonstration.* The radius *AC* (*fig.* 50) being equal to the radius *EO*, and the arc *AMD* equal to the arc *ENG*; the chord *AD* will be equal to the chord *EG*.

For, the diameter *AB* being equal to the diameter *EF*, the semicircle *AMDB* may be applied exactly to the semicircle *ENGF*, and then the curved line *AMDB* will coincide entirely with the curved line *ENGF*; but the portion *AMD* being supposed equal to the portion *ENG*, the point *D* will fall upon *G*; therefore the chord *AD* is equal to the chord *EG*.

Conversely, *AC* being supposed equal to *EO*, if the chord $AD = EG$, the arc *AMD* will be equal to the arc *ENG*.

For, if the radii *CD*, *OG*, be drawn, the two triangles *ACD*, *EOG*, will have the three sides of the one equal to the three sides of the other, each to each, namely, $AC = EO$, $CD = OG$ and $AD = EG$; therefore these triangles are equal (43); hence the angle $ACD = EOG$. Now, if the semicircle *ADB* be placed upon *EGF*, because the angle $ACD = EOG$, it is evident, that the radius *CD* will fall upon the radius *OG*, and the point *D* upon *G*, therefore the arc *AMD* is equal to the arc *ENG*.

THEOREM.

103. *In the same circle, or in equal circles, if the arc be less than half a circumference, the greater arc is subtended by the greater chord; and, conversely, the greater chord is subtended by the greater arc.*

Fig. 50. *Demonstration.* Let the arc *AH* (*fig.* 50) be greater than *AD*, and let the chords *AD* and *AH*, and the radii *CD*, *CH*, be drawn. The two sides, *AC*, *CH*, of the triangle *ACH*, are equal to the two sides *AC*, *CD*, of the triangle *ACD*, and the angle *ACH* is greater than *ACD*; hence the third side *AH* is greater than the third side *AD* (42), therefore the greater arc is subtended by the greater chord.

Conversely, if the chord *AH* be greater than *AD*, it may be inferred from the same triangles that the angle *ACH* is greater than *ACD*, and that thus the arc *AH* is greater than *AD*.

104. *Scholium.* The arcs of which we have been speaking, are supposed to be less than a semicircumference; if they were greater, the contrary would be true; in this case, as the arc increases, the chord would diminish, and the reverse; thus, the arc *AKBD* being greater than *AKBH*, the chord *AD* of the first is less than the chord *AH* of the second.

THEOREM.

105. *The radius* CG (fig. 51), *perpendicular to a chord* AB, *bisects this chord and the arc subtended by it* AGB. Fig. 51.

Demonstration. Draw the radii *CA*, *CB*; these radii are, with respect to the perpendicular *CD*, two equal oblique lines, therefore they are equally distant from the perpendicular (52), and *AD* = *DB*.

Again, since *AD* = *BD*, and *CG* is a perpendicular erected upon the middle of *AB*, each point in *CG* is at equal distances from *A* and *B* (55). The point *G* is one of these points; therefore *AG* = *GB*. But, if the chord *AG* is equal to the chord *GB*, the arc *AG* will be equal to the arc *GB* (102); therefore the radius *CG*, perpendicular to the chord *AB*, bisects the arc subtended by this chord in the point *G*.

106. *Scholium.* The centre C, the middle *D* of the chord *AB*, and the middle *G* of the arc subtended by this chord, are three points situated in the same straight line perpendicular to the chord. Now, two points in a straight line are sufficient to determine its position; therefore a straight line which passes through any two of these points must necessarily pass through the third; and must be perpendicular to the chord.

It follows also, that a perpendicular erected upon the middle of a chord passes through the centre, and the middle of the arc subtended by that chord.

For this perpendicular is the same as that let fall from the centre upon the same chord, since they both pass through the middle of the chord (51).

THEOREM.

107. *The circumference of a circle may be made to pass through any three points,* A, B, C (fig. 52), *which are not in the same* Fig. 52.

straight line, but the circumference of only one circle may be made to pass through the same points.

Demonstration. Join *AB*, *BC*, and bisect these two straight lines by the perpendiculars *DE*, *FG*; these perpendiculars will meet in a point *O*.

For the lines *DE*, *FG*, will necessarily cut each other, if they are not parallel. Let us suppose that they are parallel; the line *AB* perpendicular to *DE* will be perpendicular to *FG* (65), and the angle *K* will be a right angle; but *BK*, which is *BD* produced, is different from *BF*, since the three points *A*, *B*, *C*, are not in the same straight line; there are then two perpendiculars *BF*, *BK*, let fall from the same point upon the same straight line, which is impossible (50); therefore the perpendiculars *DE*, *FG*, will always cut each other in some point *O*.

Now the point *O*, considered with reference to the perpendicular *DE*, is at equal distances from the two points *A* and *B* (55); also this same point *O*, considered with reference to the perpendicular *FG*, is at equal distances from the two points *B* and *C*; hence the three distances *OA*, *OB*, *OC*, are equal; therefore the circumference, described from the centre *O* with the radius *OB*, will pass through the three points *A*, *B*, *C*.

It is thus proved, that the circumference of a circle may be made to pass through any three given points, which are not in the same straight line; it remains to show, that there is only one circle, which can be so described.

If there were another circle, the circumference of which passed through the three given points *A*, *B*, *C*, its centre could not be without the line *DE* (55), since, in this case, it would be at unequal distances from *A* and *B*; neither can it be without the line *FG*, for a similar reason; it will then be in both of these lines at the same time. But two lines can cut each other in only one point (32); there is therefore only one circle, whose circumference can pass through three given points.

108. *Corollary.* Two circumferences can meet each other only in two points; for, if they had three points common, they would have the same centre, and would make one and the same circumference.

THEOREM.

109. *Two equal chords are at the same distance from the centre, and of two unequal chords the less is at the greater distance from the centre.*

Demonstration 1. Let the chord $AB = DE$ (*fig.* 53). Bisect these chords by the perpendicular CF, CG, and draw the radii CA, CD. Fig. 53.

The right-angled triangles CAF, DCG, have the hypothenuses CA, CD, equal; moreover the side AF, the half of AB, is equal to the side DG, the half of DE; the triangles then are equal (59), and consequently the third side CF is equal to the third side CG; therefore the two equal chords AB, DE, are at the same distance from the centre.

2. Let the chord AH be greater than DE, the arc AKH will be greater than the arc DME (103). Upon the arc AKH take the part $ANB = DME$, draw the chord AB, and let fall the perpendicular CF upon this chord, and the perpendicular CI upon AH; CF is evidently greater than CO, and CO than CI (52); for a still stronger reason $CF > CI$. But $CF = CG$, since the chords AB, DE, are equal. Therefore $CG > CI$, and of two unequal chords the less is at the greater distance from the centre.

THEOREM.

110. *The perpendicular* BD (fig. 54), *at the extremity of the radius* AC, *is a tangent to the circumference.* Fig. 54.

Demonstration. Since every oblique line CE is greater than the perpendicular CA (52), the point E is without the circle, and the line BD has only the point A in common with the circumference; therefore BD is a tangent (97).

111. *Scholium.* We can draw through a given point A only one tangent AD to the circumference; for, if we could draw another, it would not be a perpendicular to the radius CA, and with respect to this new tangent the radius CA would be an oblique line, and the perpendicular let fall from the centre upon this tangent would be less than CA; therefore this supposed tangent would pass into the circle and become a secant.

THEOREM.

Fig. 55. 112. *Two parallels* AB, DE (fig. 55), *intercept upon the circumference equal arcs* MN, PQ.

Demonstration. The proposition admits of three cases.

1. If the two parallels are secants, draw the radius *CH* perpendicular to the chord *MP*, it will also be perpendicular to its parallel *NQ* (64), and the point *H* will be at the same time the middle of the arc *MHP* and of *NHQ* (105); whence the arc $MH = HP$, and the arc $NH = HQ$; also

$$MH - NH = HP - HQ, \text{ that is, } MN = PQ.$$

Fig. 56. 2. If of the two parallels *AB*, *DE* (*fig.* 56), one be a secant and the other a tangent; to the point of contact *H* draw the radius *CH*; this radius will be perpendicular to the tangent *DE* (110), and also to its parallel *MP*. But, since *CH* is perpendicular to the chord *MP*, the point *H* is the middle of the arc *MHP*; therefore the arcs *MH*, *HP*, comprehended between the parallels *AB*. *DE*, are equal.

3. If the two parallels, *DE*, *IL*, are tangents, the one at *H* and the other at *K*; draw the parallel secant *AB*, and we shall have, according to what has just been demonstrated $MH = HP$, and $MK = KP$; therefore the entire arc $HMK = HPK$, and it is moreover evident, that each of these arcs is a semicircumference,

THEOREM.

113. *If the circumferences of two circles cut each other in two points, the line which passes through their centres will be perpendicular to the chord, which joins the points of intersection, and will bisect it.*

Fig. 57, 58. *Demonstration.* The line *AB* (*fig.* 57, 58), which joins the points of intersection, is a chord common to the two circles; and, if a perpendicular be erected upon the middle of this chord, it must pass through each of the centres *C* and *D* (106). But through two given points only one straight line can be drawn; therefore the straight line, which passes through the centres, will be perpendicular to the middle of the common chord.

THEOREM.

114. *If the distance of two centres is less than the sum of the radii, and if at the same time the greater radius is less than the sum of the smaller and the distance of the centres, the two circles will cut each other.*

Demonstration. In order that the intersection may take place, the triangle ACD (*fig.* 57, 58) must be possible. It is necessary Fig. 57, 58. then, not only that CD (*fig.* 57) should be less than $AC + AD$, Fig. 57. but also that the greater radius AD (*fig.* 58) should be less than Fig. 58. $AC + CD$. Now, while the triangle CAD can be constructed, it is clear that the circumferences described from the centres C and D will cut each other in A and B.

THEOREM.

115. *If the distance* CD (fig. 59) *of the centres of two circles is* Fig. 59 *equal to the sum of their radii* CA, CD, *these two circles will touch each other externally.*

Demonstration. It is evident that they will have the point A common, but they can have no other, for in order that there may be two points common, it is necessary that the distance of the centres should be less than the sum of the radii (114).

THEOREM.

116. *If the distance* CD *of the centres of two circles is equal to the difference of their radii* CA, AD (fig. 60), *these two circles will* Fig. 60. *touch each other internally.*

Demonstration. In the first place it is evident, that they will have the point A common; and they can have no other, for in order that they may have two points common, it is necessary that the greater radius AD should be less than the sum of the radius AC and the distance of the centres CD (114), which is contrary to the supposition.

117. *Corollary.* Hence, if two circles touch each other, either internally or externally, the centres and the point of contact are in the same straight line.

118. *Scholium.* All the circles, which have their centres in the straight line CD and whose circumferences pass through the point A, touch each other, and have only the point A common. And if through the point A we draw AE perpendicular to CD, the straight line AE will be a tangent common to all these circles.

THEOREM.

119. *In the same circle, or in equal circles, equal angles* ACB,
Fig. 61. DCE (fig. 61), *the vertices of which are at the centre, intercept upon the circumference equal arcs* AB, DE.

Reciprocally, if the arcs AB, DE, *are equal, the angles* ACB, DCE, *also will be equal.*

Demonstration. 1. If the angle *ACB* is equal to the angle *DCE*, these two angles may be placed the one upon the other, and as their sides are equal, it is evident, that the point *A* will fall upon *D*, and the point *B* upon *E*. But in this case the arc *AB* must also fall upon the arc *DE*; for if the two arcs were not coincident, there would be points in the one or the other at unequal distances from the centre, which is impossible; therefore the arc $AB = DE$.

2. If we suppose $AB = DE$, the angle *ACB* will be equal to *DCE*; for, if these angles are not equal, let *ACB* be the greater, and let *ACI* be taken equal to *DCE*; and we have, according to what has just been demonstrated, $AI = DE$. But, by hypothesis, the arc $AB = DE$; we should consequently have $AI = AB$, or the part equal to the whole, which is impossible; therefore the angle $ACB = DCE$.

THEOREM.

120. *In the same circle, or in equal circles, if two angles at the*
Fig. 62. *centre* ACB, DCE (fig. 62), *are to each other, as two entire numbers, the intercepted arcs* AB, DE, *will be to each other, as the same numbers, and we shall have this proportion;*

angle ACB : *angle* DCE : : *arc* AB : *arc* DE.

Demonstration. Let us suppose, for example, that the angles *ACB*, *DCE*, are to each other, as 7 to 4; or, which amounts to the same, that the angle *M*, which will serve as a common measure, is contained seven times in the angle *ACB*, and four times in the angle *DCE*. The partial angles *ACm*, *mCn*, *nCp*, &c., *DCx*, *xCy*, &c., being equal to each other, the partial arcs *Am*, *mn*, *np*, &c., *Dx*, *xy*, &c., will also be equal to each other (119), and the entire arc *AB* will be to the entire arc *DE*, as 7 to 4. Now it is evident, that the same reasoning might be used, whatever numbers were substituted in the place of 7 and 4; therefore, if the ratio of the angles *ACB*, *DCE*, can be expressed by entire numbers, the arcs *AB*, *DE*, will be to each other, as the angles *ACB*, *DCE*.

121. *Scholium.* Reciprocally, if the arcs *AB*, *DE*, are to each other, as two entire numbers, the angles *ACB*, *DCE*, will be to each other, as the same numbers, and we shall have always *ACB* : *DCE* : : *AB* : *DE* ; for the partial arcs *Am*, *mn*, &c., *Dx*, *xy*, &c., being equal, the partial angles *ACm*, *mCn*, &c., *DCx*, *xCy*, &c., are also equal.

THEOREM.

122. *Whatever may be the ratio of two angles* ACB, ACD, (fig. 63), *these two angles will always be to each other, as the arcs* AB, AD, *intercepted between their sides and described from their vertices, as centres, with equal radii.* Fig. 63.

Demonstration. Let us suppose the less angle placed in the greater; if the proposition enunciated be not true, the angle *ACB* will be to the angle *ACD*, as the arc *AB* is to an arc greater or less than *AD*. Let this arc be supposed to be greater, and let it be represented by *AO* ; we shall have,

angle *ACB* : angle *ACD* : : arc *AB* : arc *AO*.

Let us now imagine the arc *AB* to be divided into equal parts, of which each shall be less than *DO*, there will be at least one point of division between *D* and *O* ; let *I* be this point, and join *CI* ; the arcs *AB*, *AI*, will be to each other, as two entire numbers, and we shall have, by the preceding theorem,

angle *ACB* : angle *ACI* : : arc *AB* : arc *AI*.

Comparing these two proportions together, and observing, that the antecedents are the same, we conclude that the consequents are proportional (III)†, namely,

angle *ACD* : angle *ACI* : : arc *AO* : arc *AI*.

But the arc *AO* is greater than the arc *AI* ; it is necessary then, in order that this proportion may take place, that the angle *ACD* should be greater than the angle *ACI* ; but it is less; it is therefore impossible, that the angle *ACB* should be to the angle *ACD*, as the arc *AB* is to an arc greater than *AD*.

By a process of reasoning altogether similar, it may be shown, that the fourth term of the proportion cannot be less than *AD* ;

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† The reference by Roman numerals is to the Introduction.

therefore it is exactly AD, and we have the proportion

$$\text{angle } ACB : \text{angle } ACD :: \text{arc } AB : \text{arc } AD.$$

123. *Corollary.* Since the angle at the centre of a circle and the arc intercepted between its sides have such a connexion, that when one increases or diminishes in any ratio whatever, the other increases or diminishes in the same ratio, we are authorized to establish one of these magnitudes as the measure of the other; thus we shall, in future, take the arc AB as the measure of the angle ACB. The only thing to be observed in the comparison of angles with each other is, that the arcs, which are used to measure them, must be described with equal radii. This is to be understood in the preceding propositions.

124. *Scholium.* It may seem more natural to measure a quantity by another quantity of the same kind, and upon this principle it would be convenient to refer all angles to the right angle; and thus, the right angle being the unit of measure, the acute angle would be expressed, by a number comprehended between 0 and 1, and an obtuse angle by a number between 1 and 2. But this manner of expressing angles would not be the most convenient in practice. It has been found much more simple to measure them by arcs of a circle on account of the facility of making arcs equal to given arcs and for many other reasons. Besides, if the measure of angles by the arcs of a circle be in some degree indirect, it is not the less easy to obtain, by means of them, the direct and absolute measure; for, if we compare the arc, which is used as the measure of an angle, with the fourth part of the circumference, we have the ratio of the given angle to a right angle, which is the absolute measure.

125. *Scholium* II. All that has been demonstrated in the three preceding propositions, for the comparison of angles with arcs, is equally applicable to the purpose of comparing sectors with arcs; for sectors are equal, when their arcs are equal, and in general they are proportional to the angles; hence *two sectors* ACB, ACD, *taken in the same circle or in equal circles, are to each other, as the arcs* AB, AD, *the bases of these sectors.*

It will be perceived therefore, that the arcs of a circle, which are used as a measure of angles, will also serve as the measure of different sectors of the same circle or of equal circles.

THEOREM.

126. *The inscribed angle* BAD (fig. 64, 65), *has for its measure the half of the arc* BD *comprehended between its sides.* Fig. 64, 65.

Demonstration. Let us suppose, in the first place, that the centre of the circle is situated in the angle *BAD* (*fig.* 64); we draw the diameter *AE*, and the radii *CB*, *CD*. The angle *BCE*, being the exterior angle of the triangle *ABC*, is equal to the sum of the two opposite interior angles, *CAB*, *ABC*. But, the triangle *BAC* being isosceles, the angle $CAB = ABC$; hence the angle *BCE* is double of *BAC*. The angle *BCE*, having its vertex at the centre, has for its measure the arc *BE*; therefore the angle *BAC* has for its measure the half of *BE*. For a similar reason the angle *CAD* has for its measure the half of *ED*; therefore $BAC + CAD$, or *BAD*, has for its measure the half $BE + ED$, or the half of *BD*. Fig. 64.

Let us suppose, in the second place, that the centre *C* (*fig.* 65), is situated without the angle *BAD*; then, the diameter *AE* being drawn, the angle *BAE* will have for its measure the half of *BE*, and the angle *DAE* the half of *DE*; hence their difference *BAD* will have for its measure the half of *BE* minus the half of *ED*, or the half of *BD*. Fig. 65.

Therefore every inscribed angle has for its measure the half of the arc comprehended between its sides.

127. *Corollary* I. All the angles *BAC*, *BDC* (*fig.* 66), &c., inscribed in the same segment, are equal; for they have each for their measure the half of the same arc *BOC*. Fig. 66.

128. *Corollary* II. Every angle *BAD* (*fig.* 67), inscribed in a semicircle, is a right angle; for it has for its measure the half of the semicircumference *BOD*, or the fourth of the circumference. Fig. 67.

To demonstrate the same thing in another way, draw the radius *AC*; the triangle *BAC* is isosceles, and the angle

$$BAC = ABC;$$

the triangle *CAD* is also isosceles, and the angle $CAD = ADC$; hence $BAC + CAD$, or $BAD = ABD + ADB$. But, if the two angles *B* and *D* of the triangle *ABD* are together equal to the third *BAD*, the three angles of the triangle will be equal to twice the angle *BAD*; they are also equal to two right angles; therefore the angle *BAD* is a right angle.

Fig. 66. 129. *Corollary* III. Every angle BAC (*fig.* 66), inscribed in a segment greater than a semicircle, is an acute angle; for it has for its measure the half of the arc BOC less than a semicircumference.

And every angle BOC, inscribed in a segment less than a semicircle, is an obtuse angle; for it has for its measure the half of the arc BAC greater than a semicircumference.

Fig. 68. 130. *Corollary* IV. The opposite angles A and C (*fig.* 68) of an inscribed quadrilateral $ABCD$ are together equal to two right angles; for the angle BAD has for its measure the half of the arc BCD, and the angle BCD has for its measure the half of the arc BAD; hence the two angles BAD, BCD, taken together, have for their measure the half of the circumference; therefore their sum is equal to two right angles.

THEOREM.

Fig. 69. 131. *The angle* BAC (fig. 69), *formed by a tangent and a chord, has for its measure the half of the arc* AMDC, *comprehended between its sides.*

Demonstration. At the point of contact A draw the diameter AD; the angle BAD is a right angle (110), and has for its measure the half of the semicircumference AMD; the angle DAC has for its measure the half of DC; therefore $BAD + DAC$, or BAC, has for its measure the half of AMD plus the half of DC, or the half of the whole arc $AMDC$.

It may be demonstrated, in like manner, that CAE has for its measure the half of the AC, comprehended between its sides.

Problems relating to the two first Sections.

PROBLEM.

Fig. 70. 132. *To divide a given straight line* AB (fig. 70) *into two equal parts.*

Solution. From the points A and B, as centres, and with a radius greater than the half of AB, describe two arcs cutting each other in D; the point D will be equally distant from the points A and B; find in like manner, either above or below the line AB a second point E equally distant from the points A and

B; through the two points *D* and *E* draw the line *DE*; this line will divide the line *AB* into two equal parts in the point *C*.

For, the two points *D* and *E* being each equally distant, from the extremities *A* and *B*, they must both be in the perpendicular which passes through the middle of *AB*. But through two given points only one straight line can be drawn; therefore the line *DE* will be this perpendicular, which divides the line *AB* into two equal parts in the point *C*.

PROBLEM.

133. *From a given point* A (fig. 71), *in the line* BC, *to erect a perpendicular to this line.* Fig. 71.

Solution. Take the points *B* and *C*, at equal distances from *A*; and from *B* and *C*, as centres, with a radius greater than *BA*, describe two arcs cutting each other in *D*; draw *AD*, which will be the perpendicular required.

For the point *D*, being equally distant from *B* and *C*, must be in a perpendicular to the middle of *BC* (55); therefore *AD* is this perpendicular.

134. *Scholium.* The same construction will serve to make a right angle *BAD* at a given point *A* in a given line *BC*.

PROBLEM.

135. *From a given point* A (fig. 72) *without the straight line* BD, *to let fall a perpendicular upon this line.* Fig. 72.

Solution. From *A*, as a centre, with a radius sufficiently great, describe an arc cutting the line *BD* in two points *B* and *D*; then find a point *E*, equally distant from the points *B* and *D* (132), and draw *AE*, which will be the perpendicular required.

For the two points *A* and *E* are each equally distant from the points *B* and *D*; therefore the line *AE* is perpendicular to the middle of *BD*.

PROBLEM.

136. *At a given point* A (fig. 73) *in the line* AB, *to make an angle equal to a given angle* K. Fig. 73.

Solution. From the vertex *K*, as a centre, with any radius, describe an arc *IL* meeting the sides of the angle, and from the point *A*, as a centre, with the same radius, describe an indefinite arc *BO*; from *B*, as a centre, with a radius equal to the chord *LI*, describe an arc cutting the arc *BO* in *D*; draw *AD*, and the angle *DAB* will be equal to the given angle *K*.

For the arcs *BD*, *LI*, have equal radii and equal chords; they are therefore equal (102), and the angle *BAD* = *IKL*.

PROBLEM.

137. *To bisect a given arc or angle.*

Fig. 74. *Solution* 1. If it is proposed to bisect the arc *AB* (*fig.* 74); from the points *A* and *B*, as centres, with the same radius, describe two arcs intersecting each other in *D*; through the point *D* and the centre *C* draw *CD*, which will divide the arc *AB* into two equal parts in the point *E*.

For, since the points *C* and *D* are each equally distant from the extremities *A* and *B* of the chord *AB*, the line *CD* is perpendicular to the middle of this chord; therefore it bisects the arc *AB* (105).

2. If it is proposed to bisect the angle *ACB*; from the vertex *C*, as a centre, describe the arc *AB*, and complete the construction, as above described. It is evident that the line *CD* will bisect the angle *ACB*.

138. *Scholium.* By the same construction, we may bisect each of the halves *AE*, *EB*, and thus, by successive subdivisions, we may divide an angle or arc into four, eight, sixteen, &c., equal parts.

PROBLEM.

Fig. 75. 139. *Through a given point* A (fig. 75), *to draw a straight line parallel to a given straight line* BC.

Solution. From the point *A*, as a centre, with a radius sufficiently great, describe the indefinite arc *EO*; from the point *E*, as a centre, with the same radius, describe the arc *AF*; take

$$ED = AF,$$

and draw *AD*, which will be the parallel required.

For, *AE* being joined, the alternate angles *AEF*, *EAD*, are equal; therefore *AD*, *EF*, are parallel (67).

PROBLEM.

Fig. 76. 140. *Two angles* A *and* B (fig. 76) *of a triangle being given, to find the third.*

Solution. Draw the indefinite line *DEF*; at the point *E* make the angle *DEC* = *A*, and the angle *CEH* = *B*; the remaining angle *HEF* will be the third angle required; for these three angles are together equal to two right angles.

PROBLEM.

141. *Two sides of a triangle* B *and* C (fig. 77) *being given, and the angle* A *contained by them, to construct the triangle.* Fig. 77.

Solution. Draw the indefinite line *DE*, and make at the point *D* the angle *EDF* equal to the given angle *A*; then take *DG* = *B*, *DH* = *C*, and draw *GH*; *DGH* will be the triangle required (36).

PROBLEM.

142. *One side and two angles of a triangle being given, to construct the triangle.*

Solution. The two given angles will be either both adjacent to the given side, or one adjacent and the other opposite. In this last case, find the third angle (140); we shall thus have the two adjacent angles. Then draw the straight line *DE* (*fig.* 78) Fig. 78. equal to the given side, at the point *D* make the angle *EDF* equal to one of the adjacent angles, and at the point *E* the angle *DEG* equal to the other; the two lines *DF*, *EG*, will cut each other in *H*, and *DEH* will be the triangle required (38).

PROBLEM.

143. *The three sides* A, B, C (fig. 79), *of a triangle being given, to construct the triangle.* Fig. 79.

Solution. Draw *DE* equal to the side *A*; from the point *E*, as a centre, with a radius equal to the second side *B*, describe an arc; from the point *D*, as a centre, with a radius equal to the third side *C*, describe another arc cutting the former in *F*; draw *DF*, *EF*, and *DEF* will be the triangle required (41).

144. *Scholium.* If one of the sides be greater than the sum of the other two, the arcs will not cut each other; but the solution will always be possible, when each side is less than the sum of other two.

PROBLEM.

145. *Two sides* A *and* B *of a triangle being given with the angle* C *opposite to the side* B, *to construct the triangle.*

Solution. The problem admits of two cases. 1. If the angle Fig. 80. *C* (*fig.* 80) is a right angle, or an obtuse angle, make the angle *EDF* equal to the angle *C*; take *DE* = *A*, from the point *E*, as a centre, and with a radius equal to the given side *B*, describe an arc cutting the line *DF* in *F*; draw *EF*, and *DEF* will be the triangle required.

It is necessary, in this case, that the side *B* should be greater than *A*, for the angle *C* being a right or an obtuse angle, it is the greatest of the angles of the triangle, and the side opposite must consequently be the greatest of the sides.

Fig. 81. 2. If the angle *C* (*fig.* 81) is acute, and *B* greater than *A*, the construction is the same, and *DEF* is the triangle required.

Fig. 82. But if, while *C* (*fig.* 82) is acute, the side *B* is less than *A*, then the arc described from the centre *E* with the radius *EF* = *B*, will cut the side *DF* in two points *F* and *G* situated on the same side of *D*; there are therefore two triangles *DEF*, *DEG*, which equally answer the conditions of the problem.

146. *Scholium.* The problem would be in every case impossible, if the side *B* were less than the perpendicular let fall from *E* upon the line *DF*.

PROBLEM.

Fig. 83. 147. *The adjacent sides* A *and* B (fig. 83) *of a parallelogram being given together with the included angle* C, *to construct the parallelogram.*

Solution. Draw the line *DE* = *A*; make the angle *FDE* = *C*, and take *DF* = *B*; describe two arcs, one from the point *F*, as a centre, with the radius *FG* = *DE*, and the other from the point *E*, as a centre, with the radius *EG* = *DF*; to the point *G*, where the two arcs cut each other, draw *FG*, *EG*; and *DEGF* will be the parallelogram required.

For, by construction, the opposite sides are equal, therefore the figure is a parallelogram (86), and it is formed with the given adjacent sides and included angle.

148. *Corollary.* If the given angle be a right angle, the figure will be a rectangle; and, if the adjacent sides are also equal, the figure will be a square.

PROBLEM.

149. *To find the centre of a given circle, or of a given arc.*

Solution. Take at pleasure three points *A*, *B*, *C* (*fig.* 84), in Fig. 84. the circumference of the circle or in the given arc; join *AB* and *BC*, and bisect them by the perpendiculars *DE*, *FG*; the point *O*, in which these perpendiculars meet, is the centre sought.

150. *Scholium.* By the same construction a circle may be found, the circumference of which will pass through three given points *A*, *B*, *C*, or in which a given triangle *ABC* may be inscribed.

PROBLEM.

151. *Through a given point, to draw a tangent to a given circle.*

Solution. If the given point *A* (*fig.* 85) be in the circumfer- Fig. 85. ence, draw the radius *CA*, and through *A* draw *AD* perpendicular to *CA*, then *AD* will be the tangent sought (110). If the point *A* (*fig.* 86) be without the circle, join the point *A* and the centre Fig 86. by the straight line *AC*; bisect *AC* in *O*, and from *O*, as a centre, with the radius *OC*, describe an arc cutting the given circle in the point *B*; draw *AB*, and *AB* will be the tangent required.

For, if we draw *CB*, the angle *CBA* inscribed in a semicircle is a right angle (128); therefore *AB*, being a perpendicular at the extremity of the radius *CB*, is a tangent.

152. *Scholium.* The point *A* being without the circle, it is evident that there are always two equal tangents *AB*, *AD*, which pass through the point *A*; they are equal (56) because the right-angled triangles *CBA*, *CDA*, have the hypothenuse *CA* common, and the side $CB = CD$; therefore $AD = AB$, and at the same time the angle $CAD = CAB$.

PROBLEM.

153. *To inscribe a circle in a given triangle* ABC (fig. 87). Fig. 87.

Bisect the angles *A* and *B* of the triangle by the straight lines *AO* and *BO*, which will meet each other in *O*; from the point *O* draw the perpendiculars *OD*, *OE*, *OF*, to the three sides of the triangle; these lines will be equal to each other. For, by construction, the angle $DAO = OAF$, and the right angle $ADO = AFO$;

consequently the third angle *AOD* is equal to the third *AOF*. Besides, the side *AO* is common to the two triangles *AOD*, *AOF*; therefore, a side and the adjacent angles of the one being respectively equal to a side and the adjacent angles of the other, the two triangles are equal; hence $DO = OF$. It may be shown, in like manner, that the two triangles *BOD*, *BOE*, are equal; consequently $OD = OE$; therefore the three perpendiculars *OD*, *OE*, *OF*, are equal to each other.

Now, if from the point *O*, as a centre, and with the radius *OD*, we describe a circle, it is evident that this circle will be inscribed in the triangle *ABC*; for the side *AB*, perpendicular to the radius at its extremity, is a tangent. The same may be said of the sides *BC*, *AC*.

154. *Scholium.* The three lines, which bisect the three angles of a triangle, meet in the same point.

PROBLEM.

Fig. 88, 89. 155. *Upon a given straight line* AB (fig. 88, 89) *to describe a segment capable of containing a given angle* C, *that is a segment such, that each of the angles, which may be inscribed in it, shall be equal to a given angle* C.

Solution. Produce *AB* toward *D*, make at the point *B* the angle $DBE = C$, draw *BO* perpendicular to *BE*, and *GO* perpendicular to *AB*, *G* being the middle of *AB*; from the point of meeting *O*, as a centre, and with the radius *OB*, describe a circle; the segment required will be *AMB*.

For, since *BF* is perpendicular to the radius at its extremity, *BF* is a tangent, and the angle *ABF* has for its measure the half of the arc *AKB* (131); besides, the angle *AMB*, as an inscribed angle, has also for its measure the half of the arc *AKB*; consequently the angle $AMB = ABF = EBD = C$; therefore each of the angles inscribed in the segment *AMB* is equal to the given angle *C*.

156. *Scholium.* If the given angle were a right angle, the segment sought would be a semicircle described upon the diameter *AB*.

PROBLEM.

157. *To find the numerical ratio of two given straight lines* AB, CD (fig. 90), *provided, however, these two lines have a common measure.* Fig. 90.

Solution. Apply the smaller *CD* to the greater *AB*, as many times as it will admit of, for example, twice with a remainder *BE*.

Apply the remainder *BE* to the line *CD*, as many times as it will admit of, for example, once with a remainder *DF*.

Apply the second remainder *DF* to the first *BE*, as many times as it will admit of, once, for example, with a remainder *BG*.

Apply the third remainder *BG* to the second *DF*, as many times as it will admit of.

Proceed thus, till a remainder arises, which is exactly contained a certain number of times in the preceding.

This last remainder will be the common measure of the two proposed lines; and, by regarding it as unity, the values of the preceding remainders are easily found, and, at length, those of the proposed lines from which their ratio in numbers is deduced.

If, for example, we find that *GB* is contained exactly twice in *FD*, *GB* will be the common measure of the two proposed lines. Let $GB = 1$, we have $FD = 2$; but *EB* contains *FD* once plus *GB*; therefore $EB = 3$; *CD* contains *EB* once plus *FD*; therefore $CD = 5$; *AB* contains *CD* twice plus *EB*; therefore $AB = 13$; consequently the ratio of the two lines *AB*, *CD*, is as 13 to 5. If the line *CD* be considered as unity, the line *AB* would be $\frac{13}{5}$; and, if the line *AB* be considered as unity, the line *CD* would be $\frac{5}{13}$.

158. *Scholium.* The method, now explained, is the same as that given in arithmetic for finding the common divisor of two numbers (*Arith.* 61), and does not require another demonstration.

It is possible, that, however far we continue the operation, we may never arrive at a remainder, which shall be exactly contained a certain number of times in the preceding. In this case the two lines have no common measure, and they are said to be *incommensurable.* We shall see, hereafter, an example of this in the ratio of the diagonal to the side of a square. But, although the exact ratio cannot be found in numbers, by neglecting the last remainder we may find an approximate ratio to a greater or less degree of exactness, according as the operation is more or less extended.

PROBLEM.

Fig. 91. 159. *Two angles* A *and* B (fig. 91) *being given, to find their common measure, if they have one, and from this their ratio in numbers.*

Solution. Describe, with equal radii, the arcs *CD*, *EF*, which may be regarded as the measure of these angles; in order then to compare the arc *CD*, *EF*, proceed as in the preceding problem; for an arc may be applied to an arc of the same radius, as a straight line is applied to a straight line. We shall thus obtain a common measure of the arcs *CD*, *EF*, if they have one, and their ratio in numbers. This ratio will be the same as that of the given angles (122); if *DO* is the common measure of the arcs, *DAO* will be the common measure of the angles.

160. *Scholium.* We may thus find the absolute value of an angle by comparing the arc, which serves as its measure, with the whole circumference. If, for example, the arc *CD* is to the circumference as 3 to 25, the angle *A* will be $\frac{3}{25}$ of four right angles, or $\frac{12}{25}$ of one right angle.

It may happen, as we have seen with respect to straight lines, that arcs also, which are compared, have not a common measure; we can then obtain, for the angles, only an approximate ratio in numbers, more or less exact, according to the degree to which the operation is extended.

SECTION THIRD.

Of the Proportions of Figures.

DEFINITIONS.

161. I SHALL call those figures *equivalent* whose surfaces are equal.

Two figures may be equivalent, however dissimilar; thus a circle may be equivalent to a square, a triangle to a rectangle, &c.

The denomination of *equal* figures will be restricted to those, which being applied, the one to the other, coincide entirely; thus two circles having the same radius are equal; and two triangles having the three sides of the one equal to the three sides of the other, each to each, are also equal.

162. Two figures are *similar*, which have the angles of the one equal to the angles of the other, each to each, and the *homologous sides* proportional. By homologous sides are to be understood those, which have the same position in the two figures, or which are adjacent to equal angles. The angles, which are equal in the two figures, are called *homologous angles.*

Equal figures are always similar, but similar figures may be very unequal.

163. In two different circles, *similar arcs*, *similar sectors*, *similar segments*, are such as correspond to equal angles at the centre. Thus, the angle *A* (*fig.* 92) being equal to the angle *O*, the arc *BC* is similar to the arc *DE*, the sector *ABC* to the sector *ODE*, &c. Fig. 92.

164. The *altitude* of a parallelogram is the perpendicular which measures the distance between the opposite sides *AB*, *CD* (*fig.* 93), considered as bases. Fig. 93.

The *altitude of a triangle* is the perpendicular *AD* (*fig.* 94), let fall from the vertex of an angle *A* to the opposite side taken for a base. Fig. 94.

The *altitude of a trapezoid* is the perpendicular *EF* (*fig.* 95) drawn between its two parallel sides *AB*, *CD.* Fig. 95.

165. The *area* and the surface of a figure are terms nearly synonymous. Area, however, is more particularly used to denote the superficial extent of the figure considered as measured, or compared with other surfaces.

THEOREM.

166. *Parallelograms, which have equal bases and equal altitudes, are equivalent.*

Demonstration. Let *AB* (*fig.* 96) be the common base of the two parallelograms *ABCD*, *ABEF*; since they are supposed to have the same altitude, the sides *DC*, *FE*, opposite to the bases, will be situated in a line parallel to *AB* (69). Now, by the nature of a parallelogram, $AD = BC$ (84), and $AF = BE$; for the same reason, $DC = AB$, and $FE = AB$; therefore $DC = FE$. If *DC* be taken from *DE*, there will remain *CE*; and if *FE*, equal to *DC*, be taken also from *DE*, there will remain *DF*; consequently $CE = DF$. Fig. 96.

Hence the triangles *DAF*, *CBE*, have the three sides of the one equal to the sides of the other, each to each; they are therefore equal (43).

But, if from the quadrilateral *ABED* the triangle *ADF* be taken, there will remain the parallelogram *ABEF*; and, if from the same quadrilateral *ABED* the triangle *CBE*, equal to the former, be taken, there will remain the parallelogram *ABCD*; therefore the two parallelograms *ABCD*, *ABEF*, which have the same base and the same altitude, are equivalent.

Fig. 97. 167. *Corollary.* Every parallelogram *ABCD* (*fig.* 97) is equivalent to a rectangle of the same base and altitude.

THEOREM.

Fig. 98. 168. *Every triangle* ABC (fig. 98) *is half of a parallelogram* ABCD *of the same base and altitude.*

Demonstration. The triangles *ABC*, *ACD*, are equal (87), therefore each is half of the parallelogram *ABCD*.

169. *Corollary* I. A triangle *ABC* is half of a rectangle *BCEF* of the same base *BC* and the same altitude *AO*; for the rectangle *BCEF* is equivalent to the parallelogram *ABCD* (167).

170. *Corollary* II. All triangles, which have equal bases and equal altitudes, are equivalent.

THEOREM.

171. *Two rectangles which have the same altitude, are to each other as their bases.*

Fig. 99. *Demonstration.* Let *ABCD*, *AEFD* (*fig.* 99), be two rectangles, which have a common altitude *AD*; they are to each other as their bases *AB*, *AE*.

Let us suppose, in the first place, that the bases *AB*, *AE*, are commensurable, and that they are to each other, as the numbers 7 and 4, for example; if we divide *AB* into 7 equal parts, *AE* will contain four of these parts; erect at each point of division, a perpendicular to the base, we shall thus form seven partial rectangles which will be equal to each other, since they will have the same base and the same altitude (166). The rectangle *ABCD* will contain seven partial rectangles, while *AEFD* will contain four; therefore the rectangle *ABCD* is to the rectangle *AEFD*, as 7 is to 4, or as *AB* is to *AE*. The same reasoning may be applied to any other ratio beside that of 7 to 4; hence, whatever be the ratio, provided it is commensurable, we have

$$ABCD : AEFD :: AB : AE.$$

Let us suppose, in the second place, that the bases *AB*, *AE* (*fig.* 100), are incommensurable; we shall have notwithstanding Fig. 100.

$$ABCD : AEFD :: AB : AE.$$

For, if this proportion be not true, the three first terms remaining the same, the fourth will be greater or less than *AE*. Let us suppose that it is greater, and that we have

$$ABCD : AEFD :: AB : AO.$$

Divide the line *AB* into equal parts smaller than *EO*, and there will be at least one point of division *I* between *E* and *O*: at this point erect the perpendicular *IK*; the bases *AB*, *AI*, will be commensurable, and we shall have, according to what has just been demonstrated,

$$ABCD : AIKD :: AB : AI.$$

But we have, by hypothesis,

$$ABCD : AEFD :: AB : AO.$$

In these two proportions the antecedents are equal, therefore the consequents are proportional (III); that is

$$AIKD : AEFD :: AI : AO.$$

Now *AO* is greater than *AI*; it is necessary then, in order that this proportion may take place, that the rectangle *AEFD* should be greater than *AIKD*; but it is less; therefore the proportion is impossible, and *ABCD* cannot be to *AEFD*, as *AB* is to a line greater than *AE*.

By a process entirely similar it may be shown, that the fourth term of the proportion cannot be smaller than *AE*; consequently it is equal to *AE*.

Whatever therefore be the ratio of the bases, two rectangles *ABCD*, *AEFD*, of the same altitude, are to each other as their bases *AB*, *AE*.

THEOREM.

172. *Any two rectangles* ABCD, AEGF (fig. 101), *are to each* Fig. 101. *other, as the products of their bases by their altitudes, that is,*

$$ABCD : AEGF :: AB \times AD : AE \times AF.$$

Demonstration. Having disposed the two rectangles in such a manner, that the angles at A shall be opposite to each other, produce the sides GE, CD, till they meet in H; the two rectangles $ABCD$, $AEHD$, have the same altitude AD; they are consequently to each other as their bases AB, AE. Likewise the two rectangles $AEHD$, $AEGF$, have the same altitude AE; these are therefore to each other as their bases AD, AF. We have thus the two proportions

$$ABCD : AEHD :: AB : AE,$$
$$AEHD : AEGF :: AD : AF.$$

Multiplying these proportions in order and observing, that the connecting term $AEHD$ may be omitted, being a multiplier common to the antecedent and consequent, we have

$$ABCD : AEGF :: AB \times AD : AE \times AF.$$

173. *Scholium.* We may take for the measure of a rectangle the product of its base by its altitude, provided that, by this product, we understand that of two numbers which are the number of linear units contained in the base, and the number of linear units contained in the altitude.

This measure, however, is not absolute, but relative; it supposes that we estimate, in a similar manner, another rectangle by measuring its sides by the same linear unit; we obtain thus a second product, and the ratio of these two products is equal to that of the rectangles, conformably to the proposition, which has just been demonstrated.

Fig. 102. If, for example, the base of a rectangle A (fig. 102) be three units and its altitude ten, the rectangle would be represented by the number 3×10, or 30, a number which, thus disconnected, has no meaning; but, if we have a second rectangle B, whose base is twelve and altitude seven units, this rectangle will be represented by the number 7×12, or 84. Whence the two rectangles A and B are to each other, as 30 to 84. If therefore it is agreed to take the rectangle A, as the unit of measure for surfaces, the rectangle B will have for its absolute measure $\frac{84}{30}$, that is, it will be equal to $\frac{84}{30}$ superficial units.

The more common and simple method is to take the square as the unit of surface; and that square has been preferred, whose side is the unit of length; the measure therefore, which we have regarded as simply relative, becomes absolute. The number 30, for example, by which we have measured the rectangle A, represents 30 superficial units, or 30 of those squares, the side of each of which is equal to unity. This is illustrated by figure 102.

In geometry, the product of two lines often signifies the same thing as their *rectangle*, and this expression is introduced into arithmetic to denote the product of two unequal numbers, as that of *square* is used to express the product of a number by itself.

The squares of the numbers 1, 2, 3, &c., are 1, 4, 9, &c. Thus a double line gives a quadruple square (*fig.* 103), a triple Fig. 103. line a square nine times as great, and so on.

THEOREM.

174. *The area of any parallelogram is equal to the product of its base by its altitude.*

Demonstration. The parallelogram $ABCD$ (*fig.* 97) is equiva- Fig. 97. lent to the rectangle $ABEF$, which has the same base AB and the same altitude BE (167); but this last has for its measure $AB \times BE$ (173); therefore $AB \times BE$ is equal to the area of the parallelogram $ABCD$.

175. *Corollary.* Parallelograms of the same base are to each other as their altitudes, and parallelograms of the same altitude are to each other as their bases; for, A, B, C, being any three magnitudes whatever, we have generally $A \times C : B \times C :: A : B$.

THEOREM.

176. *The area of a triangle is equal to the product of its base by half of its altitude.*

Demonstration. The triangle ABC (*fig.* 104) is half of the Fig. 104. parallelogram $ABCE$, which has the same base BC and the same altitude AD (168); now the area of the parallelogram $= BC \times AD$ (174); therefore the area of the triangle $= \frac{1}{2} BC \times AD$, or $BC \times \frac{1}{2} AD$.

177. *Corollary.* Two triangles of the same altitude are to each other as their bases, and two triangles of the same base are to each other as their altitudes.

THEOREM.

178. *The area of a trapezoid* ABCD (fig. 105) *is equal to the* Fig. 105. *product of its altitude* EF *by half of the sum of its parallel sides* AB, CD.

Demonstration. Through the point I, the middle of the side CB, draw KL parallel to the opposite side AD, and produce DC till it meet KL in K.

In the triangles IBL, ICK, the side $IB = IC$, by construction; the angle $LIB = CIK$, and the angle $IBL = ICK$, since CK and BL are parallel (67); therefore these triangles are equal (38), and the trapezoid $ABCD$ is equivalent to the parallelogram $ADKL$, and has for its measure $EF \times AL$.

But $AL = DK$; and, since the triangle IBL is equal to the triangle KCI, the side $BL = CK$; therefore

$$AB + CD = AL + DK = 2AL;$$

thus AL is half the sum of the sides AB, CD; and consequently the area of the trapezoid $ABCD$ is equal to the product of the altitude EF by half the sum of the sides AB, CD, which may be expressed in this manner; $ABCD = EF \times \left(\frac{AB + CD}{2}\right)$.

179. *Scholium.* If through the point I, the middle of BC, IH be drawn parallel to the base AB, the point H will also be the middle of AD; for the figure $AHIL$ is a parallelogram, as well as $DHIK$, since the opposite sides are parallel; we have therefore $AH = IL$, and $DH = IK$; but $IL = IK$, because the triangles BIL, CIK, are equal; therefore $AH = DH$.

It may be remarked, that the line $HI = AL = \frac{AB + CD}{2}$; therefore the area of the trapezoid may be expressed also by $EF \times HI$; that is, it is equal to the product of the altitude of the trapezoid by the line joining the middle points of the sides which are not parallel.

THEOREM.

Fig. 106. 180. *If a line* AC (fig. 106) *is divided into two parts* AB, BC, *the square described upon the whole line* AC *will contain the square described upon the part* AB, *plus the square described upon the other part* BC, *plus twice the rectangle contained by the two parts* AB, BC; *which may be thus expressed,*

$$\overline{AC}^2 \text{ or } (AB + BC)^2 = \overline{AB}^2 + \overline{BC}^2 + 2AB \times BC.$$

Demonstration. Construct the square $ACDE$, take $AF = AB$, draw FG parallel to AC, and BH parallel to AE.

The square *ACDE* is divided into four parts; the first *ABIF* is the square described upon *AB*, since *AF* was taken equal to *AB*; the second *IGDH* is the square described upon *BC*; for, since $AC = AE$, and $AB = AF$, the difference $AC - AB = AE - AF$, which gives $BC = EF$; but, on account of the parallels, $IG = BC$, and $DG = EF$, therefore *HIGD* is equal to the square described upon *BC*. These two parts being taken from the whole square, there remain the two rectangles *BCGI*, *EFIH*, which have each for their measure $AB \times BC$; therefore the square described upon *AC*, &c.

181. *Scholium.* This proposition corresponds to that given in algebra for the formation of the square of a binomial, which is thus expressed,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

THEOREM.

182. *If the line* AC (fig. 107) *is the difference of two lines* AB, BC, *the square described upon* AC *will contain the square of* AB, *plus the square of* BC, *minus twice the rectangle contained by* AB *and* BC; *that is,* $\overline{AC}^2$ *or* $(AB - BC)^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,AB \times BC$. Fig. 107.

Demonstration. Construct the square *ABIF*, take $AE = AC$, draw *CG* parallel to *BI*, *HK* parallel to *AB*, and finish the square *EFLK*.

The two rectangles *CBIG*, *GLKD*, have each for their measure $AB \times BC$; if we subtract them from the whole figure *ABILKEA*, which has for its value $\overline{AB}^2 + \overline{BC}^2$ it is evident, that there will remain the square *ACDE*; therefore, if the line *AC*, &c.

183. *Scholium.* This proposition answers to the algebraic formula $(a - b)^2 = a^2 + b^2 - 2ab$.

THEOREM.

184. *The rectangle contained by the sum and difference of two lines is equal to the difference of their squares; that is,*

$$(AB + BC) \times (AB - BC) = \overline{AB}^2 - \overline{BC}^2 \text{ (fig. 108).}$$ Fig. 108.

Demonstration. Construct upon *AB* and *AC* the squares *ABIF*, *ACDE*; produce *AB* making $BK = BC$, and complete the rectangle *AKLE*.

The base *AK* of the rectangle is the sum of the two lines *AB*, *BC*, its altitude *AE* is the difference of these lines; therefore the rectangle $AKLE = (AB + BC) \times (AB - BC)$. But this same rectangle is composed of two parts $ABHE + BHLK$, and the part *BHLK* is equal to the rectangle *EDGF*, for $BH = DE$, and $BK = EF$; consequently $AKLE = ABHE + EDGF$. Now these two parts form the square *ABIF*, minus the square *DHIG* which is the square described upon *BC*; therefore

$$(AB + BC) \times (AB - BC) = \overline{AB}^2 - \overline{BC}^2.$$

185. *Scholium.* This proposition agrees with the algebraic formula $(a + b) \times (a - b) = (a^2 - b^2)$ (Alg. 34).

THEOREM.

186. *The square described upon the hypothenuse of a right-angled triangle is equal to the sum of the squares described upon the two other sides.*

Fig. 109. *Demonstration.* Let *ABC* (*fig.* 109) be a triangle right-angled at *A*. Having constructed squares upon the three sides, let fall, from the right angle upon the hypothenuse, the perpendicular *AD*, which produce to *E*, and draw the diagonals *AF*, *CH*.

The angle *ABF* is composed of the angle *ABC* plus the right angle *CBF*; and the angle *HBC* is composed of the same angle *ABC* plus the right angle *ABH*; hence the angle $ABF = HBC$. But $AB = BH$, being sides of the same square; and $BF = BC$, for the same reason; consequently the triangles *ABF*, *HBC*, have two sides and the included angle of the one respectively equal to two sides and the included angle of the other; they are therefore equal (36).

The triangle *ABF* is half of the rectangle *BE*†, which has the same base *BF* and the same altitude *BD* (169). Also the triangle *HBC* is half of the square *AH*; for, the angle *BAC* being a right angle as well as *BAL*, *AC* and *AL* are in the same straight line parallel to *HB*; hence the triangle *HBC* and the square *AH* have the same base *BH*, and the same altitude *AB*; therefore the triangle is half of the square.

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† An abridged expression for *BDEF*.

It has already been proved, that the triangle *ABF* is equal to the triangle *HBC*; consequently the rectangle *BDEF*, double of the triangle *ABF*, is equivalent to the square *AH*, double of the triangle *HBC*. It may be demonstrated, in the same manner, that the rectangle *CDEG* is equivalent to the square *AI*; but the two rectangles *BDEF*, *CDEG*, taken together, make the square *BCGF*; therefore the square *BCGF*, described upon the hypothenuse, is equal to the sum of the squares *ABHL*, *ACIK*, described upon the two other sides; or, $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$.

187. *Corollary* I. The square of one of the sides of a right-angled triangle is equal to the square of the hypothenuse minus the square of the other side; or $\overline{AB}^2 = \overline{BC}^2 - \overline{AC}^2$.

188. *Corollary* II. Let *ABCD* (*fig.* 118) be a square, *AC* its Fig. 118. diagonal; the triangle *ABC* being right-angled and isosceles, we have $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 2\overline{AB}^2$; therefore *the square described upon the diagonal* AC *is double of the square described upon the side* AB.

This property may be rendered sensible by drawing, through the points *A* and *C*, parallels to *BD*, and through the points *B* and *D*, parallels to *AC*; a new square *EFGH* is thus formed which is the square of *AC*. It is manifest that *EFGH* contains eight triangles, each of which is equal to *ABE*, and that *ABCD* contains four of them; therefore the square *EFGH* is double of *ABCD*.

Since $\overline{AC}^2 : \overline{AB}^2 :: 2 : 1$, we have, by extracting the square root, $AC : AB :: \sqrt{2} : 1$; therefore *the diagonal of a square is incommensurable with its side* (Alg. 99).

This will be more fully developed hereafter.

189. *Corollary* III. It has been demonstrated, that the square *AH* (*fig.* 109) is equivalent to the rectangle *BDEF*; now, on Fig. 109. account of the common altitude *BF*, the square *BCGF* is to the rectangle *BDEF* as the base *BC* is to the base *BD*; therefore

$$\overline{BC}^2 : \overline{AB}^2 :: BC : BD,$$

or, *the square of the hypothenuse is to the square of one of the sides of the right angle, as the hypothenuse is to the segment adjacent to this side.* We give the name ~~of *segment*~~ to that part of the hypothenuse cut off by the perpendicular let fall from the right angle; thus *BD* is the segment adjacent to the side *AB*, and *DC* the segment adjacent to the side *AC*. We have likewise

$$\overline{BC}^2 : \overline{AC}^2 :: BC : CD.$$

190. *Corollary* IV. The rectangles $BDEF$, $DCGE$, having also the same altitude DE, are to each other as their bases BD, CD. Now these rectangles are equivalent to the squares AH, AI, therefore,

$$\overline{AB}^2 : \overline{AC}^2 :: BD : DC,$$

or, *the squares of the two sides of a right angle are to each other, as the segments of the hypothenuse adjacent to these sides.*

THEOREM.

Fig. 110. 191. *In a triangle* ABC (fig. 110), *if the angle* C *be acute, the square of the side opposite to it will be less than the sum of the squares of the sides containing it, and,* AD *being drawn perpendicular to* BC, *the difference will be equal to double the rectangle* BC × CD, *or,*

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2BC \times CD.$$

Demonstration. The proposition admits of two cases. 1. If the perpendicular fall within the triangle ABC, we shall have $BD = BC - CD$, and consequently (182)

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2BC \times CD;$$

adding $\overline{AD}^2$ to each member, we have

$$\overline{AD}^2 + \overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 + \overline{AD}^2 - 2BC \times CD;$$

but the right-angled triangles ABD, ADC, give $\overline{AD}^2 + \overline{BD}^2 = \overline{AB}^2$, $\overline{CD}^2 + \overline{AD}^2 = \overline{AC}^2$; therefore

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD.$$

2. If the perpendicular AD fall without the triangle ABC, we shall have $BD = CD - BC$, and consequently (182)

$$\overline{BD}^2 = \overline{CD}^2 + \overline{BC}^2 - 2BC \times CD;$$

add to each $\overline{AD}^2$, and we shall obtain, as before,

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD.$$

THEOREM.

Fig. 111. 192. *In a triangle* ABC (fig. 111), *if the angle* C *be obtuse, the square of the side opposite to it will be greater than the sum of the* *squares of the sides containing it, and,* AD *being drawn perpendicular to* BC *produced, the difference will be equal to double the rectangle* BC × CD, *or,*

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 + 2BC \times CD.$$

Demonstration. The perpendicular cannot fall within the triangle; for if it should fall, for example, upon E, the triangle ACE would have at the same time a right angle E and an obtuse angle C, which is impossible (75); consequently it falls without, and we have $BD = BC + CD$, and from this (180)

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 + 2BC \times CD.$$

Adding to each term $\overline{AD}^2$, and making the reductions as in the preceding theorem, we obtain

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2BC \times CD.$$

193. *Scholium.* The right-angled triangle is the only one in which the sum of the squares of two of the sides is equal to the square of the third; for, if the angle contained by their sides be acute, the sum of their squares will be greater than the square of the side opposite; if it be obtuse, the reverse will be true.

THEOREM.

194. *In any triangle* ABC (fig. 112), *if we draw from the vertex to the middle of the base the line* AE, *we shall have* Fig. 112.

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{AE}^2 + 2\overline{EB}^2.$$

Demonstration. Let fall the perpendicular AD upon the base BC, the triangle AEC will give (191),

$$\overline{AC}^2 = \overline{AE}^2 + \overline{EC}^2 - 2EC \times ED;$$

the triangle ABE will give (192),

$$\overline{AB}^2 = \overline{AE}^2 + \overline{EB}^2 + 2EB \times ED;$$

therefore, by adding the corresponding members, and observing that $EB = EC$, we shall have

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{AE}^2 + 2\overline{EB}^2.$$

195. *Corollary. In every parallelogram the sum of the squares of the sides is equal to the sum of the squares of the diagonals.*

For the diagonals AC, BD (*fig.* 113), mutually bisect each other in the point E (88), and the triangle ABC gives Fig. 113.

$$\overline{AB}^2+\overline{BC}^2=2\overline{AE}^2+2\overline{BE}^2;$$

the triangle ADC gives likewise

$$\overline{AD}^2+\overline{DC}^2=2\overline{AE}^2+2\overline{DE}^2;$$

adding the corresponding members and observing that $BE=DE$, we have

$$\overline{AB}^2+\overline{AD}^2+\overline{DC}^2+\overline{BC}^2=4\overline{AE}^2+4\overline{DE}^2.$$

But $4\overline{AE}^2$ is the square of $2AE$ or of AC; and $4\overline{DE}^2$ is the square of BD; therefore the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

THEOREM.

Fig. 114. 196. *The line* DE (fig. 114), *drawn parallel to the base of a triangle* ABC, *divides the sides* AB, AC, *proportionally; so that* AD : DB : : AE : EC.

Demonstration. Join BE and DC; the two triangles BDE, DEC, have the same base DE; they have also the same altitude, since the vertices B and C are situated in a parallel to the base; therefore the triangles are equivalent (170).

The triangles ADE, BDE, of which the common vertex is E, have the same altitude, and are to each other as their bases AD, DB (177); thus,

$$ADE : BDE :: AD : DB.$$

The triangles ADE, DEC, of which the common vertex is D, have also the same altitude, and are to each other as their bases AE, EC; that is, $ADE : DEC :: AE : EC$.

But it has been shown that the triangle $BDE=DEC$; therefore, on account of the common ratio in the two proportions (III),

$$AD : DB :: AE : EC.$$

197. *Corollary* I. We obtain from the above theorem by *composition* (IV)

$$AD+DB : AD :: AE+EC : AE,$$

or $$AB : AD :: AC : AE,$$

also $$AB : BD :: AC : CE.$$

198. *Corollary* II. *If between two straight lines* AB, CD,

Fig. 115. (fig. 115), *parallels* AC, EF, GH, BD, *&c., be drawn, these two straight lines will be cut proportionally, and we shall have,*

AE : CF : : EG : FH : : GB : HD.

For, let O be the point of meeting of the straight lines, AB, CD; in the triangle OEF, the line AC being drawn parallel to the base EF, $OE : AE :: OF : CF$, or $OE : OF :: AE : CF$.

In the triangle OGH we have likewise

$$OE : EG :: OF : FH, \text{ or } OE : OF :: EG : FH;$$

therefore, on account of the common ratio $OE : OF$, these two proportions give

$$AE : CF :: EG : FH.$$

It may be demonstrated, in the same manner, that

$$EG : FH :: GB : HD,$$

and so on; therefore the lines AB, CD, are cut proportionally by the parallels EF, GH, &c.

THEOREM.

199. *Reciprocally, if the sides* AB, AC (fig. 116), *are cut proportionally by the line* DE, *so that* AD : DB :: AE : EC, *the line* DE *will be parallel to the base* BC. Fig. 116.

Demonstration. If DE is not parallel to BC let us suppose that DO is parallel to it; then, according to the preceding theorem, $AD : DB :: AO : OC$.

But, by hypothesis, $AD : DB :: AE : EC$;

consequently $AO : OC :: AE : EC$,

which is impossible, since of the antecedents AE is greater than AO, and of the consequents EC is less than OC; hence the line, drawn through the point D parallel to BC, does not differ from DE; therefore DE is this line.

200. *Scholium.* The same conclusion might be deduced from the proportion $AB : AD :: AC : AE$.

For this proportion would give (IV)

$$AB - AD : AD :: AC - AE : AE, \text{ or } BD : AD :: EC : AE.$$

THEOREM.

201. *The line* AD (fig. 117), *which bisects the angle* BAC *of a triangle, divides the base* BC *into two segments* BD, DC, *proportional to the adjacent sides* AB, AC; *so that,* BD : DC :: AB : AC. Fig. 117.

Demonstration. Through the point C draw CE parallel to AD to meet BA produced.

In the triangle BCE, the line AD being parallel to the base 196), GE, $BD : DC :: AB : AE$.

But the triangle ACE is isosceles; for, on account of the parallels AD, CE, the angle $ACE = DAC$, and the angle $AEC = BAD$ (67); and, by hypothesis, $DAC = BAD$; therefore the angle $ACE = AEC$, and consequently $AE = AC$ (48); substituting then AC for AE in the preceding proportion, we have

$$BD : DC :: AB : AC.$$

THEOREM.

202. *Two equiangular triangles have their homologous sides proportional and are similar.*

Fig. 119. *Demonstration.* Let ABC, CDE (*fig.* 119), be two triangles, which have their angles equal, each to each, namely, $BAC = CDE$, $ABC = DCE$, and $ACB = DEC$; the homologous sides, or those adjacent to the equal angles, will be proportional, that is,

$$BC : CE :: BA : CD :: AC : DE.$$

Let the homologous sides BC, CE, be in the same straight line, and produce the sides BA, ED, till they meet in F.

Since BCE is a straight line, and the angle $BCA = CED$, it follows that AC is parallel to DE (67). Also, since the angle $ABC = DCE$, the line AB is parallel to DC; therefore the figure $ACDF$ is a parallelogram.

In the triangle BFE, the line AC being parallel to the base FE, $BC : CE :: BA : AF$ (196); substituting in the place of AF its equal CD, we have

$$BC : CE :: BA : CD.$$

In the same triangle BFE, BF being considered as the base, since CD is parallel to BF, $BC : CE :: FD : DE$. Substituting for FD its equal AC, we have

$$BC : CE :: AC : DE.$$

From these two proportions, which contain the same ratio $BC : CE$, we have

$$AC : DE :: BA : CD.$$

Hence the equiangular triangles BAC, CDE, have the homologous sides proportional. But two figures are similar, when they have, at the same time, their angles equal, each to each, and the homologous sides proportional (162); therefore the equiangular triangles BAC, CDE, are two similar figures.

203. *Corollary.* In order to be similar, it is sufficient that two triangles have two angles of the one respectively equal to two angles of the other; for then the third angles will be equal and the two triangles will be equiangular.

204. *Scholium.* It may be remarked, that in similar triangles the homologous sides are opposite to equal angles; thus, the angle ACB being equal to DEC, the side AB is homologous to DC; likewise AC, DE, are homologous, being opposite to the equal angles ABC, DCE. Knowing the homologous sides, we readily form the proportions;

$$AB : DC :: AC : DE :: BC : CE.$$

THEOREM.

205. *Two triangles, which have their homologous sides proportional, are equiangular and similar.*

Demonstration. Let us suppose that

$$BC : EF :: AB : DE :: AC : DF \text{ (fig. 120)};$$ Fig. 120.

the triangles ABC, DEF, will have their angles equal, namely, $A = D$, $B = E$, $C = F$.

Make, at the point E, the angle $FEG = B$, and at the point F, the angle $EFG = C$, the third angle G will be equal to the third angle A, and the two triangles ABC, EFG, will be equiangular; whence, by the preceding theorem, $BC : EF :: AB : EG$; but, by hypothesis, $BC : EF :: AB : DE$; consequently $EG = DE$. We have, moreover, by the same theorem, $BC : EF :: AC : FG$; but, by hypothesis, $BC : EF :: AC : DF$; consequently $FG = DF$; hence the triangles EGF, DEF, have the three sides of the one equal to the three sides of the other, each to each; they are therefore equal (43). But, by construction, the triangle EGF is equiangular with the triangle ABC; therefore the triangles DEF, ABC, are, in like manner, equiangular and similar.

206. *Scholium.* It will be perceived, by the two last propositions, that, when the angles of one triangle are respectively equal to those of another, the sides of the former are proportional to those of the latter, and the reverse; so that one of these conditions is sufficient to establish the similitude of triangles. This is not true of figures having more than three sides; for, with respect to those of only four sides, or quadrilaterals, we may alter the proportion of the sides without changing the angles, or change the angles without altering the sides; thus, from the angles being equal it does not follow that the sides are proportional, or the reverse. We see, for example, that by drawing EF (*fig.* 121) Fig. 121. parallel to BC, the angles of the quadrilateral $AEFD$ are equal

to those of the quadrilateral *ABCD*; but the proportion of the sides is different. Also, without changing the four sides *AB*, *BC*, *CD*, *AD*, we can bring the points *B* and *D* nearer together, or remove them further apart, which would alter the angles.

207. *Scholium.* The two preceding theorems (202, 205), which, properly speaking, make only one, added to that of the square of the hypothenuse (186), are of all the propositions of geometry the most remarkable for their importance, and the number of results that are derived from them; they are almost sufficient of themselves, for all applications and for the resolution of all problems; the reason is, that all figures may be resolved into triangles, and any triangle whatever into two right-angled triangles. Thus the general properties of triangles involve those of all figures.

THEOREM.

208. *Two triangles, which have an angle of the one equal to an angle of the other and the sides about these angles proportional, are similar.*

Fig. 122. *Demonstration.* Let the angle $A = D$ (*fig.* 122), and let $AB : DE :: AC : DF$, the triangle *ABC* is similar to the triangle *DEF*.

Take $AG = DE$, and draw *GH* parallel to *BC*, the angle $AGH = ABC$ (67); and the triangle *AGH* will be equiangular with the triangle *ABC*;

whence $AB : AG :: AC : AH$;
but, by hypothesis, $AB : DE :: AC : DF$,
and, by construction, $AG = DE$; therefore $AH = DF$. The two triangles *AGH*, *DEF*, have the two sides and the included angle of the one respectively equal to two sides and the included angle of the other; they are consequently equal. But the triangle *AGH* is similar to *ABC*; therefore *DEF* is also similar to *ABC*.

THEOREM.

209. *Two triangles, which have the sides of the one parallel, or which have them perpendicular, to those of the other, each to each, are similar.*

Fig. 123. *Demonstration.* 1. If the side *AB* (*fig.* 123) is parallel to *DE*, and *BC* to *EF*, the angle *ABC* will be equal to *DEF* (70); if

moreover, *AC* is parallel to *DF*, the angle *ACB* will be equal to *DFE*, and also *BAC* to *EDF*; therefore the triangles *ABC*, *DEF*, are equiangular and consequently similar.

2. Let the side *DE* (*fig.* 124) be perpendicular to *AB*, and the side *DF* to *AC*. In the quadrilateral *AIDH* the two angles *I*, *H*, will be right angles, and the four angles will be together equal to four right angles (80); therefore the two remaining angles *IAH*, *IDH*, are together equal to two right angles. But the two angles *EDF*, *IDH*, are together equal to two right angles, consequently the angle *EDF* is equal to *IAH* or *BAC*. In like manner, if the third side *EF* is perpendicular to the third side *BC*, it may be shown that the angle *DFE* = *C*, and *DEF* = *B*; therefore the two triangles *ABC*, *DEF*, which have the sides of the one perpendicular to those of the other, each to each, are equiangular and similar. Fig. 124.

210. *Scholium.* In the first of the above cases the homologous sides are the parallel sides, and in the second the homologous sides are those which are perpendicular to each other. Thus, in the second case, *DE* is homologous to *AB*, *DF* to *AC*, and *EF* to *BC*.

The case of the perpendicular sides admits of the two triangles being differently situated from those represented in figure 124; but the equality of the respective angles may always be proved, either by means of quadrilaterals, such as *AIDH*, which have two right angles, or by comparing two triangles which, beside the vertical angles, have each a right angle; or we can always suppose, within the triangle *ABC*, a triangle *DEF*, the sides of which shall be parallel to those of the triangle to be compared with *ABC*, and then the demonstration will be the same as that given for the case of figure 124.

THEOREM.

211. *Lines* AF, AG, &c. (fig. 125), *drawn at pleasure through the vertex of a triangle, divide proportionally the base* BC *and its parallel* DE, *so that* Fig. 125.

DI : BF : : IK : FG : : KL : GH, *&c.*

Demonstration. Since *DI* is parallel to *BF*, the triangles *ADI*, *ABF*, are equiangular, and *DI* : *BF* : : *AI* : *AF*; also, *IK* being parallel to *FG*, *AI* : *AF* : : *IK* : *FG*; hence, on account of the common ratio, *AI* : *AF*, *DI* : *BF* :: *IK* : *FG*. It may be shown, in like manner, that *IK* : *FG* :: *KL* : *GH*, &c.; therefore the line *DE* is divided at the points *I*, *K*, *L*, as the base *BC* is at the points *F*, *G*, *H*.

212. *Corollary.* If *BC* should be divided into equal parts at the points *F*, *G*, *H*, the parallel *DE* would be divided likewise into equal parts at the points *I*, *K*, *L*.

THEOREM.

Fig. 126. 213. *If from the right angle* A (fig. 126) *of a right-angled triangle the perpendicular* AD *be let fall upon the hypothenuse;*

1. *The two partial triangles* ABD, ADC, *will be similar to each other and to the whole triangle* ABC;

2. *Each side* AB *or* AC *will be a mean proportional between the hypothenuse* BC *and the adjacent segment* BD *or* DC;

3. *The perpendicular* AD *will be a mean proportional between the two segments* BD, DC.

Demonstration. 1. The triangles *BAD*, *BAC*, have the angle *B* common; moreover the right angle *BDA* = *BAC*; consequently the third angle *BAD* of the one is equal to the third angle *C* of the other, and the two triangles are equiangular and similar. It may be demonstrated, in the same manner, that the triangle *DAC* is similar to the triangle *BAC*; therefore the three triangles are equiangular and similar.

2. Since the triangle *BAD* is similar to the triangle *BAC*, their homologous sides are proportional. Now the side *BD* in the smaller triangle is homologous to the side *BA* in the larger, because they are opposite to the equal angles, *BAD*, *BCA*; the hypothenuse *BA* of the smaller is homologous to the hypothenuse *BC* of the larger;

hence $$BD : BA :: BA : BC.$$

In the same manner it may be shown that

$$DC : AC :: AC : BC;$$

therefore each of the sides *AB*, *AC*, is a mean proportional between the hypothenuse and the segment adjacent to this side.

3. By comparing the homologous sides of the similar triangles *ABD*, *ADC*, we have

$$BD : AD :: AD : DC;$$

therefore the perpendicular *AD* is a mean proportional between the segments *BD*, *DC*, of the hypothenuse.

214. *Scholium.* The proportion $BD : AB :: AB : BC$, by putting the product of the extremes equal to that of the means, gives

$$\overline{AB}^2 = BD \times BC.$$

We have, in line manner,

$$\overline{AC}^2 = DC \times BC,$$

hence $\overline{AB}^2 + \overline{AC}^2 = BD \times BC + DC \times BC;$
the second member, otherwise expressed, is $(BD + DC) \times BC$, or $\overline{BC}^2;$

consequently $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2;$
therefore the square of the hypothenuse BC is equal to the sum of the squares of the two other sides AB, AC. We thus fall again upon the proposition of the square of the hypothenuse by a process very different from that before pursued; from which it appears, that, properly speaking, the proposition of the square of the hypothenuse is a consequence of the proportionality of the sides of equiangular triangles. Thus the fundamental propositions of geometry reduce themselves, as it were, to this single one, that equiangular triangles have their homologous sides proportional.

It often happens, as in the present instance, that by pursuing the consequences of one or several propositions, we return to the propositions before demonstrated. Generally speaking, that which particularly characterizes the theorems of geometry, and which is an irresistible proof of their certainty, is, that by combining them together in any manner whatever, provided the reasoning be just, we always fall upon accurate results. This would not be the case, if any proposition were false, or only true to a certain degree; it would often happen, that, by combining the propositions together, the error would augment and become sensible. We have examples of this in all those demonstrations, in which we make use of the *reductio ad absurdum.* These demonstrations, in which the object is to prove that two quantities are equal, consist in making it evident, that if there were between them the least inequality, we should be led by a course of reasoning to a manifest and palpable absurdity; whence we are obliged to conclude that the two quantities are equal.

Fig. 127. 215. *Corollary.* If from the point A (*fig.* 127) of the circumference of a circle two chords AB, AC, be drawn to the extremities of the diameter BC, the triangle ABC will be right-angled at A (128); whence, 1. *the perpendicular* AD *is a mean proportional between the segments* BD, DC, *of the diameter*, or, which amounts to the same thing,

$$\overline{AD}^2 = BD \times DC.$$

2. *The chord* AB *is a mean proportional between the diameter* BC *and the adjacent segment* BD;

or, $$\overline{AB}^2 = BD \times BC.$$

Also $\overline{AC}^2 = DC \times BC$; therefore $\overline{AB}^2 : \overline{AC}^2 :: BD : DC$. If we compare $\overline{AB}^2$ with $\overline{BC}^2$, we shall have

$$\overline{AB}^2 : \overline{BC}^2 :: BD : BC;$$

we have, in like manner,

$$\overline{AC}^2 : \overline{BC}^2 :: DC : BC.$$

These ratios of the squares of the sides to each other and to the square of the hypothenuse have already been given in articles 189, 190.

THEOREM.

216. *Two triangles, which have an angle in the one equal to an angle in the other, are to each other as the rectangles of the sides* Fig. 128. *which contain the equal angles; thus, the triangle* ABC (fig. 128) *is to the triangle* ADE, *as the rectangle* AB × AC *is to the rectangle* AD × AE.

Demonstration. Draw BE; the two triangles ABE, ADE, whose common vertex is E, have the same altitude, and are to each other as their bases AB, AD (177); hence

$$ABE : ADE :: AB : AD.$$

In like manner,

$$ABC : ABE :: AC : AE;$$

multiplying the two proportions in order and omitting the common term ABE, we have,

$$ABC : ADE :: AB \times AC : AD \times AE.$$

217. *Corollary.* The two triangles would be equivalent, if the rectangle $AB \times AC$ were equal to the rectangle $AD \times AE$, or if

$AB : AD :: AE : AC$, which is the case when the line DC is parallel to *BE.*

THEOREM.

218. *Two similar triangles are to each other as the squares of their homologous sides.*

Demonstration. Let the angle $A = D$ (*fig.* 122), and the angle $B = E$, then, by the preceding proposition, Fig. 122.

$$ABC : DEF :: AB \times AC : DE \times DF;$$

and, because the triangles are similar,

$$AB : DE :: AC : DF.$$

This proportion being multiplied in order by the identical proportion.

$$AC : DF :: AC : DF,$$

we shall have

$$AB \times AC : DE \times DF :: \overline{AC}^2 : \overline{DF}^2.$$

Hence

$$ABC : DEF :: \overline{AC}^2 : \overline{DF}^2.$$

Therefore two similar triangles *ABC*, *DEF*, are to each other as the squares of the homologous sides *AC*, *DF*, or as the squares of any other two homologous sides.

THEOREM.

219. *Two similar polygons are composed of the same number of triangles, which are similar to each other and similarly disposed.*

Demonstration. In the polygon *ABCDE* (*fig.* 129) draw from an angle *A* the diagonals *AC*, *AD*, to the other angles. In the other polygon *FGHIK* draw, in like manner, from the angle *F*, homologous to *A*, the diagonals *FH*, *FI*, to the other angles. Fig. 129.

Since the polygons are similar, the angle *ABC* is equal to the homologous angle *FGH* (162), moreover the sides *AB*, *BC*, are proportional to the sides *FG*, *GH*, so that

$$AB : FG :: BC : GH.$$

It follows from this, that the triangles *ABC*, *FGH*, having an angle of the one equal to an angle of the other and the sides about the equal angles proportional, are similar (208), consequently the angle $BCA = GHF$. These equal angles being subtracted from the equal angles *BCD*, *GHI*, the remaining angles ACD, FHI, will be equal. Now, since the triangles ABC, FGH, are similar,

$$AC : FH :: BC : GH;$$

besides, on account of the polygons being similar (162),

$$BC : GH :: CD : HI;$$

consequently $AC : FH :: CD : HI$;

but we have seen that the angle $ACD = FHI$; consequently the triangles ACD, FHI, have an angle of the one equal to an angle of the other and the sides about the equal angles proportional; they are therefore similar (208). We might proceed in the same manner to demonstrate, that the remaining triangles are similar, whatever be the number of the sides of the proposed polygons; therefore two similar polygons are composed of the same number of triangles, which are similar to each other and similarly disposed.

220. *Scholium.* The converse of this proposition is equally true; *if two polygons are composed of the same number of triangles, which are similar to each other and similarly disposed, these two polygons will be similar.*

For, the triangles being similar, the angles $ABC = FGH$, $BCA = GHF$, $ACD = FHI$; consequently $BCD = GHI$, also $CDE = HIK$, &c. Moreover,

$$AB : FG :: BC : GH :: AC : FH :: CD : HI, \text{ \&c.};$$

consequently the two polygons have their angles respectively equal and their sides proportional; therefore they are similar.

THEOREM.

221. *The perimeters of similar polygons are as their homologous sides, and their surfaces are as the squares of these sides.*

Demonstration. 1. By the nature of similar figures we have

Fig. 129. $AB : FG :: BC : GH :: CD : HI$, &c. (*fig.* 129),

and from this series of equal ratios we may infer, that the sum of the antecedents $AB + BC + CD$, &c., the perimeter of the first figure is to the sum of the consequents $FG + GH + HI$, &c., the perimeter of the second figure, as one antecedent, is to its consequent (IV), or as the side AB is to its homologous side FG.

2. The triangles ABC, FGH, being similar

$$ABC : FGH :: \overline{AC}^2 : \overline{FH}^2 \quad (218);$$

in like manner, ACD, FHI, being similar,

$$ACD : FHI :: \overline{AC}^2 : \overline{FH}^2;$$

hence, on account of the common ratio $\overline{AC}^2 : \overline{FH}^2$,

$$ABC : FGH :: ACD : FHI.$$

By a similar process of reasoning it may be shown that

$$ACD : FHI :: ADE : FIK;$$

and so on, if there should be a greater number of triangles. Hence, from this series of equal ratios, the sum of the antecedents $ABC + ACD + ADE$, or the polygon $ABCDE$, is to the sum of consequents $FGH + FHI + FIK$, or the polygon $FGHIK$, as one antecedent ABC is to its consequent FGH, or as $\overline{AB}^2$ is to $\overline{FG}^2$ (219). Therefore the surfaces of similar polygons are to each other, as the squares of their homologous sides.

222. *Corollary.* If three similar figures be constructed whose homologous sides are equal to the three sides of a right-angled triangle, the figure described upon the greatest side will be equal to the sum of the two others; for the three figures will be proportional to the squares of their homologous sides; now the square of the hypothenuse is equal to the sum of the squares of the two other sides; therefore, &c.

THEOREM.

223. *The parts of two chords which cut each other in a circle are reciprocally proportional; that is,* AO : DO :: CO : OB (fig. 130). Fig. 130.

Demonstration. Join AC and BD. In the triangles ACO, BOD, the angles at O are equal, being vertical angles, and the angle A is equal to the angle D, because they are inscribed in the same segment (127); for the same reason the angle $C = B$; therefore these triangles are similar, and the homologous sides give the proportion

$$AO : DO :: CO : OB.$$

224. *Corollary.* Hence $AO \times OB = DO \times CO$; therefore the rectangle of the two parts of one of the chords is equal to the rectangle of the two parts of the other.

THEOREM.

225. *If from a point* O (fig. 131), *taken without a circle, secants* OB, OC, *be drawn terminating in the concave arc* BC, *the entire* Fig. 131.

secants will be reciprocally proportional to the parts without the circle; that is, OB : OC : : OD : OA.

Demonstration. Join AC and BD. The triangles OAC, OBD, have the angle O common; moreover the angle $B = C$ (126); therefore the triangles are similar; and the homologous sides give the proposition

$$OB : OC :: OD : OA.$$

226. *Corollary.* The rectangle $OA \times OB = OC \times OD$.

227. *Scholium.* It may be remarked, that this proposition has great analogy with the preceding; the only difference is, that the two chords AB, CD, instead of intersecting each other in the circle, meet without it. The following proposition may also be regarded as a particular case of this.

THEOREM.

Fig. 132. 228. *If from the same point* O (fig. 132), *taken without the circle, a tangent* OA *be drawn and a secant* OC, *the tangent will be a mean proportional between the secant and the part without the circle; that is,* OC : OA : : OA : OD, *or,* $\overline{OA}^2 = OC \times OD$.

Demonstration. By joining AD and AC, the triangles OAD, OAC, have the angle O common; moreover, the angle OAD formed by a tangent and a chord (131) has for its measure the half of the arc AD, and the angle C has the same measure; consequently the angle $OAD = C$; therefore the two triangles are similar, and $OC : OA :: OA : OD$, which gives $\overline{OA}^2 = OC \times OD$.

THEOREM.

Fig. 133. 229. *In any triangle* ABC (fig. 133), *if the angle* A *be bisected by the line* AD, *the rectangle of the sides* AB, AC, *will be equal to the rectangle of the segments* BD, DC, *plus the square of the bisecting line* AD.

Demonstration. Describe a circle the circumference of which shall pass through the points A, B, C; produce AD till it meet the circumference, and join CE.

The triangle BAD is similar to the triangle EAC; for, by hypothesis, the angle $BAD = EAC$; moreover the angle $B = E$, since they have each for their measure the half of the arc AC;

consequently the triangles are similar; and the homologous sides give the proportion

$$BA : AE :: AD : AC;$$

whence $BA \times AC = AE \times AD$; but $AE = AD + DE$, and, by multiplying each by AD, we have $AE \times AD = \overline{AD}^2 + AD \times DE$; besides, $AD \times DE = BD \times DC$ (224); therefore

$$BA \times AC = \overline{AD}^2 + BD \times DC.$$

THEOREM.

230. *In every triangle* ABC (fig. 134) *the rectangle of two of the sides* AB, AC, *is equal to the rectangle contained by the diameter* CE *of the circumscribed circle and the perpendicular* AD, *let fall upon the third side* BC. Fig. 134.

Demonstration. Join AE, and the triangles ABD, AEC, are right-angled, the one at D and the other at A; moreover the angle $B = E$; consequently the triangles are similar; and they give the proportion, $AB : CE :: AD : AC$; whence

$$AB \times AC = CE \times AD.$$

231. *Corollary.* If these equal quantities be multiplied by BC, we shall have $AB \times AC \times BC = CE \times AD \times BC$. Now $AD \times BC$ is double the surface of the triangle (176); therefore *the product of the three sides of a triangle is equal to the surface multiplied by double the diameter of the circumscribed circle.*

The product of three lines is sometimes called a *solid*, for a reason that will be given hereafter. The value of this product is easily conceived by supposing the three lines reduced to numbers and these numbers multiplied together.

232. *Scholium.* It may be demonstrated also, that *the surface of a triangle is equal to its perimeter multiplied by half of the radius of the inscribed circle.*

For the triangles AOB, BOC, AOC (*fig.* 87), which have their common vertex in O, have for their common altitude the radius of the inscribed circle; consequently the sum of these triangles will be equal to the sum of the bases AB, BC, AC, multiplied by half of the radius OD; therefore the surface of the triangle ABC is equal to the product of its perimeter by half of the radius of the inscribed circle. Fig 87.

THEOREM.

Fig. 135. 233. *In every inscribed quadrilateral figure* ABCD (fig. 135), *the rectangle of the two diagonals* AC, BD, *is equal to the sum of the rectangles of the opposite sides; that is,*

$$AC \times BD = AB \times CD + AD \times BC.$$

Demonstration. Take the arc $CO = AD$, and draw BO meeting the diagonal AC in I.

The angle $ABD = CBI$, since one has for its measure half of the arc AD, and the other half of CO equal to AD. The angle $ADB = BCI$, because they are inscribed in the same segment AOB; consequently the triangle ABD is similar to the triangle IBC, and $AD : CI :: BD : BC$; whence

$$AD \times BC = CI \times BD.$$

Again, the triangle ABI is similar to the triangle BDC; for the arc AD being equal to CO, if we add to each of these OD we shall have the arc $AO = DC$; consequently the angle ABI is equal to DBC; moreover the angle $BAI = BDC$, because they are inscribed in the same segment; therefore the triangles ABI, BDC, are similar, and the homologous sides give the proportion $AB : BD :: AI : CD$; whence,

$$AB \times CD = AI \times BD.$$

Adding the two results above found, and observing that

$$AI \times BD + CI \times BD = (AI + CI) \times BD = AC \times BD,$$

we have

$$AD \times BC + AB \times CD = AC \times BD.$$

234. *Scholium.* We may demonstrate, in a similar manner, another theorem with respect to an inscribed quadrilateral figure. The triangle ABD being similar to BIC, $BD : BC :: AB : BI$, whence

$$BI \times BD = BC \times AB.$$

If we join CO, the triangle ICO, similar ABI, is similar to BDC, and gives the proportion $BD : CO :: DC : OI$, whence we have $OI \times BD = CO \times DC$, or CO being equal to AD,

$$OI \times BD = AD \times DC.$$

Adding these two results, and observing that $BI \times BD + OI \times BD$ reduces itself to $BO \times BD$, we obtain

$$BO \times BD = AB \times BC + AD \times DC.$$

If we had taken $BP = AD$, and had drawn CKP, we should have found by similar reasoning

$$CP \times CA = AB \times AD + BC \times CD.$$

But the arc BP being equal to CO, if we add to each BC, we shall have the arc $CBP = BCO$; consequently the chord CP is equal to the chord BO, and the rectangles $BO \times BD$ and $CP \times CA$, are to each other as BD is to CA; therefore

$$BD : CA :: AB \times BC + AD \times DC : AB \times AD + BC \times CD;$$

that is, *the two diagonals of an inscribed quadrilateral figure are to each other as the sums of the rectangles of the sides adjacent to their extremities.*

By means of these two theorems the diagonals may be found, when the sides are known.

THEOREM.

235. *Let* P (fig. 136) *be a given point within a circle in the radius* AC, *and let there be taken a point* Q *without the circle in the same radius produced such that* CP : CA :: CA : CQ; *if, from any point* M *of the circumference, straight lines* MP, MQ, *be drawn to the points* P *and* Q, *these straight lines will always be in the same ratio, and we shall have* MP : MQ :: AP : AQ. Fig. 136.

Demonstration. By hypothesis $CP : CA :: CA : CQ$; putting CM in the place of CA we shall have $CP : CM :: CM : CQ$; consequently the triangles CPM, CQM, having an angle of the one equal to an angle of the other and the sides about the equal angles proportional, are similar (208); therefore

$$MP : MQ :: CP : CM \text{ or } CA.$$

But the proportion

$$CP : CA :: CA : CQ$$

gives, by *division*,

$$CP : CA :: CA - CP : CQ - CA,$$

or $\quad CP : CA :: AP : AQ;$

therefore $\quad MP : MQ :: AP : AQ.$

Problems relating to the Third Section.

PROBLEM.

236. *To divide a given straight line into any number of equal parts, or into parts proportional to any given lines.*

Fig. 137. *Solution.* 1. Let it be proposed to divide the line *AB* (*fig.* 137) into five equal parts; through the extremity *A* draw the indefinite straight line *AG*, and take *AC* of any magnitude whatever, and apply it five times upon *AG*; through the last point of the division *G* draw *GB*, and through *C* draw *CI* parallel to *GB*; *AI* will be a fifth part of the line *AB*, and, by applying *AI* five times upon *AB*, the line *AB* will be divided into five equal parts.

For, since *CI* is parallel to *GB*, the sides *AG*, *AB*, are cut proportionally in *C* and *I* (196). But *AC* is a fifth part of *AG*, therefore *AI* is a fifth part of *AB*.

Fig. 138. 2. Let it be proposed to divide the line *AB* (*fig.* 138) into parts proportional to the given lines *P*, *Q*, *R*. Through the extremity *A* draw the indefinite straight line *AG*, and take $AC = P$, $CD = Q$, $DE = R$; join *EB*, and through the points *C*, *D*, draw *CI*, *DK*, parallel to *EB*; the line *AB* will be divided at *I* and *K* into parts proportional to the given lines *P*, *Q*, *R*.

For, on account of the parallels *CI*, *DK*, *EB*, the parts *AI*, *IK*, *KB*, are proportional to the parts *AC*, *CD*, *DE* (196); and, by construction, these are equal to the given lines *P*, *Q*, *R*.

PROBLEM.

237. *To find a fourth proportional to three given lines* A, B, C
Fig. 139. (fig. 139).

Solution. Draw the two indefinite lines *DE*, *DF*, making any angle with each other. On *DE* take $DA = A$, $DB = B$; and upon *DF* take $DC = C$; join *AC*, and through the point *B* draw *BX* parallel to *AC*; *DX* will be the fourth proportional required.

For, since *BX* is parallel to *AC*, $DA : DB :: DC : DX$; but the three first terms of this proportion are equal to the three given lines, therefore *DX* is the fourth proportional required.

238. *Corollary.* We might find in the same manner a third proportional to two given lines *A*, *B*; for it would be the same as the fourth proportional to the three lines *A*, *B*, *B*.

PROBLEM.

239. *To find a mean proportional between two given lines* A *and* B (fig. 140). Fig. 140.

Solution. On the indefinite line DF take $DE = A$, and $EF = B$; on the whole line DF, as a diameter, describe the semicircumference DGF; at the point E erect upon the diameter the perpendicular EG meeting the circumference in G; EG will be the mean proportional sought.

For the perpendicular GE, let fall from the point in the circumference upon the diameter, is a mean proportional between the two segments of the diameter DE, EF (215), and these two segments are equal to the two given lines A and B.

PROBLEM.

240. *To divide a given line* AB (fig. 141) *into two parts in such a manner, that the greater shall be a mean proportional between the whole line and the other part.* Fig. 141

Solution. At the extremity B of the line AB erect the perpendicular BC equal to half of AB; from the point C as a centre, and with the radius CB, describe a circle; draw AC cutting the circumference in D, and take $AF = AD$; the line AB will be divided at the point F in the manner required; that is,

$$AB : AF :: AF : FB.$$

For AB, being a perpendicular to the radius CB at its extremity CB, is a tangent; and, if AC be produced till it meet the circumference in E, we shall have

$$AE : AB :: AB : AD \quad (228),$$

and hence $AE - AB : AB :: AB - AD : AD$ (IV).

But, since the radius BC is half of AB, the diameter DE is equal to AB, and consequently $AE - AB = AD = AF$; also, since $AF = AD$, $AB - AD = FB$; therefore,

$$AF : AB :: FB : AD \text{ or } AF,$$

and by *inversion* $AB : AF :: AF : FB$.

241. *Scholium.* When a line is divided in this manner, it is said to be divided *in extreme and mean ratio.* Its application will be seen hereafter.

It may be remarked, that the secant AE is divided in extreme and mean ratio at the point D; for since $AB = DE$,

$$AE : DE :: DE : AD.$$

PROBLEM.

Fig. 142. 242. *Through a given point* A (fig. 142) *in a given angle* BCD, *to draw a line* BD *in such a manner that the parts* AB, AD, *comprehended between the point* A *and the two sides of the angle shall be equal.*

Solution. Through the point A draw AE parallel to CD, take $BE = CE$, and through the points B and A draw BAD, which will be the line required.

For AE being parallel to CD, $BE : EC :: BA : AD$, but $BE = EC$; therefore $BA = AD$.

PROBLEM.

243. *To make a square equivalent to a given parallelogram, or to a given triangle.*

Fig. 143. *Solution.* 1. Let $ABCD$ (*fig.* 143) be the given parallelogram, AB its base, and DE its altitude; between AB and DE find a mean proportional XY (239); the square described upon XY will be equivalent to the parallelogram $ABCD$.

For, by construction, $AB : XY :: XY : DE$; hence

$$\overline{XY}^2 = AB \times DE;$$

but $AB \times DE$ is the measure of the parallelogram, and $\overline{XY}^2$ is that of the square, therefore they are equivalent.

Fig. 144. 2. Let ABC (*fig.* 144) be the given triangle, BC its base, and AD its altitude; find a mean proportional between BC and half of AD, and let XY be this mean proportional; the square described upon XY will be equivalent to the triangle ABC.

For, since $BC : XY :: XY : \frac{1}{2} AD$, $\overline{XY}^2 = BC \times \frac{1}{2} AD$; therefore the square described upon XY is equivalent to the triangle ABC.

PROBLEM.

Fig. 145. 244. *Upon a given line* AD (fig. 145) *to construct a rectangle* ADEX *equivalent to a given rectangle* ABFC.

Solution. Find a fourth proportional to the three lines AD, AB, AC (137), and let AX be this fourth proportional; the rectangle contained by AD and AX will be equivalent to the rectangle $ABFC$.

For, since $AD : AB :: AC : AX$, $AD \times AX = AB \times AC$; therefore the rectangle $ADEX$ is equivalent to the rectangle $ABFC$.

PROBLEM.

245. *To find in lines the ratio of the rectangle of two given lines* A *and* B (fig. 148) *to the rectangle of two given lines* C *and* D. Fig. 148.

Solution. Let X be a fourth proportional to the three given lines B, C, D; the ratio of the two lines A and X will be equal to that of the two rectangles $A \times B$, $C \times D$.

For, since $B : C :: D : X$, $C \times D = B \times X$; therefore

$$A \times B : C \times D :: A \times B : B \times X :: A : X.$$

246. *Corollary.* In order to obtain the ratio of the squares described upon two lines A and C, find a third proportional X to the lines A and C, so that we may have the proportion

$$A : C :: C : X;$$

then we shall have $A^2 : C^2 :: A : X$.

PROBLEM.

247. *To find in lines the ratio of the product of three given lines* A, B, C (fig. 149), *to the product of three given lines* P, Q, R. Fig. 149.

Solution. Find a fourth proportional X to the three given lines P, A, B; and a fourth proportional Y to the three given lines C, Q, R. The two lines X and Y will be to each other as the products $A \times B \times C$, $P \times Q \times R$.

For, since $P : A :: B : X$, $A \times B = P \times X$; and, by multiplying each of these by C, we shall have

$$A \times B \times C = C \times P \times X.$$

In like manner, since

$$C : Q :: R : Y,\ Q \times R = C \times Y;$$

and, by multiplying each of these by P, we shall have

$$P \times Q \times R = P \times C \times Y;$$

therefore the product

$A \times B \times C$: the product $P \times Q \times R :: C \times P \times X : P \times C \times Y :: X : Y$.

PROBLEM.

248. *To make a triangle equivalent to a given polygon.*

Solution. Let $ABCDE$ (*fig.* 146) be the given polygon. Fig. 146. Draw the diagonal CE, which cuts off the triangle CDE; through the point D draw DF parallel to CE to meet AE produced; join CF, and the polygon $ABCDE$ will be equivalent to the polygon $ABCF$, which has one side less.

For the triangles CDE, CFE, have the common base CE, they are also of the same altitude, for their vertices D, F, are in a line DF parallel to the base; therefore the triangles are equivalent. Adding to each of these the figure $ABCE$ and we shall have the polygon $ABCDE$ equivalent to the polygon $ABCF$.

We can in like manner cut off the angle B by substituting for the triangle ABC the equivalent triangle AGC, and then the pentagon $ABCDE$ will be transformed into an equivalent triangle GCF.

The same process may be applied to any other figure; for, by making the number of sides one less at each step, we shall at length arrive at an equivalent triangle.

249. *Scholium.* As we have already seen, that a triangle may be transformed into an equivalent square (243), we may accordingly find a square equivalent to any given rectilineal figure; this is called *squaring* the rectilineal figure, or finding the *quadrature* of it.

The problem of the *quadrature of the circle* consists in finding a square equivalent to a circle whose diameter is given.

PROBLEM.

250. *To make a square which shall be equal to the sum or the difference of two given squares.*

Fig. 147. *Solution.* Let A and B (*fig.* 147) be the sides of the given squares.

1. If it is proposed to find a square equal to the sum of these squares, draw the two indefinite lines ED, EF, at right angles to each other; take $ED = A$ and $EG = B$; join DG, and DG will be the side of the square sought.

For the triangle DEG being right-angled, the square described upon DG will be equal to the sum of the squares described upon ED and EG.

2. If it is proposed to find a square equal to the difference of the given squares, form in like manner a right angle FEH, take GE equal to the smaller of the sides A and B; from the point G, as a centre, and with a radius GH equal to the other side, describe an arc cutting EH in H; the square described upon EH

will be equal to the difference of the squares described upon the lines A and B.

For in the right-angled triangle GEH the hypothenuse $GH = A$, and the side $GE = B$; therefore the square described upon EH is equal to the difference of the squares described upon the given sides A and B.

251. *Scholium.* We can thus find a square equal to the sum of any number of squares; for the construction by which two are reduced to one, may be used to reduce three to two and these two to one, and so of a larger number. Also a similar method may be employed when certain given squares are to be subtracted from others.

PROBLEM.

252. *To construct a square which shall be to a given square* ABCD (fig. 150), *as the line* M *is to the line* N. Fig. 150

Solution. On the indefinite line EG take $EF = M$, and $FG = N$; on EG, as a diameter, describe a semicircle, and at the point F erect upon the diameter the perpendicular FH. From the point H draw the chords HG, HE, which produce indefinitely; on the first take HK equal to the side AB of the given square, and through the point K draw KI parallel to EG; HI will be the side of the square sought.

For, on account of the parallels KI, GE,

$$HI : HK :: HE : HG;$$

hence $$\overline{HI}^2 : \overline{HK}^2 :: \overline{HE}^2 : \overline{HG}^2 \quad \text{(v)}.$$

But, in the right-angled triangle EHG,

$$\overline{HE}^2 : \overline{HG}^2 :: \text{segment } EF : \text{the segment } FG \quad (215),$$

or, as M is to N;

therefore $$\overline{HI}^2 : \overline{HK}^2 :: M : N.$$

But $HK = AB$; therefore

the square upon HI : the square upon AB :: $M : N$.

PROBLEM.

253. *Upon a side* FG (fig. 129), *homologous to* AB, *to describe a polygon similar to a given polygon* ABCDE. Fig. 129.

Solution. In the given polygon draw the diagonals AC, AD. At the point F make the angle $GFH = BAC$, and at the point G the angle $FGH = ABC$; the lines FH, GH, will cut each other in H, and the triangle FGH will be similar to ABC. Likewise upon FH, homologous to AC, construct the triangle FIH similar to ADC, and upon FI, homologous to AD, construct the triangle FIK similar to ADE. The polygon $FGHIK$ will be similar to $ABCDE$.

For these two polygons are composed of the same number of triangles, which are similar to each other and similarly disposed (219).

PROBLEM.

254. *Two similar figures being given, to construct a similar figure which shall be equal to their sum or their difference.*

Solution. Let A and B be two homologous sides of the given figures, find a square equal to the sum or the difference of the squares described upon A and B; let X be the side of this square, X will be, in the figure sought, the side homologous to A and B in the given figures. The figure may then be constructed by the preceding problem.

For similar figures are as the square of their homologous sides; but the square of the side X is equal to the sum or the difference of the squares described upon the homologous sides A and B; therefore the figure described upon the side X is equal to the sum or the difference of the similar figures described upon the sides A and B.

PROBLEM.

255. *To construct a figure similar to a given figure, and which shall be to this figure in the given ratio of* M *to* N.

Solution. Let A be a side of the given figure, and X the homologous side of the figure sought; the square of X must be to the square of A, as M is to N (221); X then may be found by art. 252; and, knowing X, we may finish the problem by art. 253.

PROBLEM.

Fig. 151. 256. *To construct a figure similar to the figure* P (fig. 151) *and equivalent to the figure* Q.

Solution. Find the side M of a square equivalent to the figure P, and the side N of a square equivalent to the figure Q. Then let X be a fourth proportional to the three given lines M, N, AB; upon the side X, homologous to AB, describe a figure similar to the figure P; it will be equivalent to the figure Q.

For, by calling Y the figure described upon the side X, we shall have

$$P : Y :: \overline{AB}^2 : X^2.$$

But, by construction,

$$AB : X :: M : N,$$

or $$\overline{AB}^2 : X^2 :: M^2 : N^2;$$

therefore $$P : Y :: M^2 : N^2.$$

We have also, by construction, $M^2 = P$, and $N^2 = Q$;
consequently $P : Y :: P : Q$;
hence $Y = Q$; therefore the figure Y is similar to the figure P and equivalent to the figure Q.

PROBLEM.

257. *To construct a rectangle equivalent to a given square* C (fig. 152), *and whose adjacent sides shall make a given sum* AB. Fig. 152.

Solution. On AB, as a diameter, describe a semicircle, and draw DE parallel to the diameter, and at a distance AD, equal to a side of the given square C. From the point E, in which the parallel cuts the circumference, let fall upon the diameter the perpendicular EF; AF and FB will be the sides of the rectangle sought.

For their sum is equal to AB, and their rectangle $AF \times FB$ is equal to the square of EF (215), or of AD; therefore the rectangle is equivalent to the given square C.

258. *Scholium.* It is necessary in order that the problem may be possible, that the distance AD should not exceed the radius, that is, that the side of the square should not exceed half of the line AB.

PROBLEM.

259. *To construct a rectangle equivalent to a square* C (fig. 153), *and whose adjacent sides shall differ by a given quantity* AB. Fig. 153.

Solution. On the given line AB, as a diameter, describe a circle; from the extremity of the diameter draw the tangent AD equal to the side of the square C. Through the point D and the centre O draw the secant DF; DE and DF will be the adjacent sides of the rectangle required.

For, 1. the difference of the sides is equal to the diameter EF or AB; 2. the rectangle $DE \times DF$ is equal to $\overline{AD}^2$ (228); therefore this rectangle will be equivalent to the given square C.

PROBLEM.

260. *To find the common measure, if there be one, between the diagonal and side of a square.*

Fig. 154. *Solution.* Let $ABCG$ (*fig.* 154) be any square, and AC its diagonal.

We are, in the first place, to apply CB to CA, as often as it can be done (157); and in order to this, let there be described, from the centre C, and with a radius CB, the semicircle DBE. It will be seen, that CB is contained once in AC with a remainder AD; the result of the first operation therefore is the quotient 1 with the remainder AD, which is to be compared with BC, or its equal AB.

We may take $AF = AD$, and apply AF actually to AB; and we should find that it is contained twice with a remainder. But, as this remainder and the following ones go on diminishing and would soon become too small to be perceived, on account of the imperfection of the mechanical operation, we can conclude nothing with regard to the question, whether the lines AC, CB, have a common measure or not. Now there is a very simple method, by which we may avoid these decreasing lines, and which only requires an operation to be performed upon lines of the same magnitude.

The angle ABC being a right angle, AB is a tangent, and AE is a secant, drawn from the same point, so that

$$AD : AB :: AB : AE \quad (228).$$

Thus, in the second operation which has for its object to compare AD with AB, we may, instead of the ratio of AD to AB, take that of AB to AE. Now AB, or its equal CD, is contained twice in AE with a remainder AD; therefore the result of the second operation is the quotient 2 with the remainder AD, which is to be compared with AB.

The third operation, which consists in comparing *AD* with *AB*, reduces itself likewise to comparing *AB*, or its equal *CD*, with *AE*, and we have still the quotient 2 with the remainder *AD*.

Whence it is evident that the operation will never terminate, and that accordingly there is no common measure between the diagonal and the side of a square, a truth already made known by a numerical operation, since these two lines are to each other :: $\sqrt{2} : 1$ (188), but which is rendered clearer by the geometrical solution.

261. It is not then possible to find in numbers the exact ratio of the diagonal to the side of a square; but we may approximate it to any degree we please by means of the continued fraction which is equal to this ratio. The first operation gave for a quotient 1; the second and each of the others continued without end gives 2; thus the fraction under consideration becomes

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \&c.$$

If, for example, we take the four first terms of this fraction, we find that its value added to the first quotient 1 is $1\frac{12}{29}$, or $\frac{41}{29}$†; so that the approximate ratio of the diagonal to the side of a square is :: 41 : 29. The ratio might be found more exactly by taking a greater number of terms.

SECTION FOURTH.

Of regular Polygons and the Measure of the Circle.

DEFINITION.

262. A POLYGON, which is at the same time equiangular and equilateral, is called a *regular polygon.*

Regular polygons admit of any number of sides. The equilateral triangle is one of three sides; and the square one of four.

THEOREM.

263. *Two regular polygons of the same number of sides are similar figures.*

Demonstration. Let there be, for example, the two regular hexagons *ABCDEF*, *abcdef* (*fig.* 155); the sum of the angles is Fig. 155.

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† See note on continued fractions at the end of Lacroix's Arithmetic.

the same in both, and is equal to eight right angles (82). The angle A is the sixth part of this sum as well as the angle a; therefore the two angles A and a are equal; the same may be said of the angles B and b, C and c, &c.

Moreover, since by the nature of these polygons the sides AB, BC, CD, &c., are equal, as also ab, bc, cd, &c.,

$$AB : ab :: BC : bc :: CD : cd, \text{&c.};$$

consequently the two figures under consideration have their angles equal and their homologous sides proportional; therefore they are similar (162).

264. *Corollary.* The perimeters of two regular polygons of the same number of sides are to each other as their homologous sides, and their surfaces are as the squares of these sides (221).

265. *Scholium.* The angle of a regular polygon is determined by the number of its sides like the angle of an equiangular polygon (79).

THEOREM.

266. *Every regular polygon may be inscribed in a circle and may be circumscribed about a circle.*

Fig. 156. *Demonstration.* Let $ABCDE$ &c. (*fig.* 156), be any regular polygon, and let there be described a circle, whose circumference shall pass through the three points A, B, C; let O be its centre, and OP a perpendicular let fall upon the middle of the side BC; join AO and OD.

The quadrilateral $OPCD$ may be placed upon the quadrilateral $OPBA$; in fact the side OP is common, and the angle $OPC = OPB$, each being a right angle, consequently the side PC will fall upon its equal PB, and the point C upon B. Moreover, by the nature of the polygon, the angle $PCD = PBA$; therefore CD will take the direction BA, and CD being equal to BA, the point D will fall upon A, and the two quadrilaterals will coincide throughout. Hence the distance OD is equal to the distance OA, and the circumference, which passes through the three points A, B, C, will pass also through the point D. By similar reasoning it may be shown, that the circumference, which passes through the three vertices B, C, D, will pass through the next vertex E, and so on; therefore the same circumference, which passes through the three points A, B, C, passes through all the vertices of the angles of the polygon, and the polygon is inscribed in this circumference.

Furthermore, with respect to this circumference, all the sides AB, BC, CD, &c. are equal chords; they are accordingly equally distant from the centre (109); if therefore from the point O, as a centre, and with the radius OP, a circle be described, the circumference will touch the side BC and all the other sides of the polygon, each at its middle point, and the circle will be inscribed in the polygon, or the polygon will be circumscribed about the circle.

267. *Scholium* I. The point O, the common centre of the inscribed and circumscribed circle, may be regarded also as the centre of the polygon; and for this reason we call the *angle of the centre* the angle AOB formed by the two radii drawn to the extremities of the same side AB.

Since all the chords AB, BC, &c., are equal, it is evident that all the angles at the centre are equal, and that the value of each is found by dividing four right angles by the number of the sides of the polygon.

268. *Scholium* II. In order to inscribe a regular polygon of a certain number of sides in a given circle, it is only necessary to divide the circumference into as many equal parts as the polygon has sides; for, the arcs being equal, the chords AB, BC, CD, &c. (*fig.* 158), will be equal; the triangles ABO, BOC, COD, &c., will also be equal, for the sides of the one will be respectively equal to those of the other; consequently all the angles ABC, BCD, CDE, &c., will be equal; therefore the figure $ABCDE$ &c. will be a regular polygon. Fig. 158.

PROBLEM.

269. *To inscribe a square in a given circle.*

Solution. Draw the diameters AC, BD (*fig.* 157), cutting each other at right angles; join the extremities A, B, C, D, and the figure $ABCD$ will be the inscribed square. Fig. 157.

For, the angles AOB, BOC, &c., being equal, the chords AB, BC, &c. are equal.

270. *Scholium.* The triangle BOC being right-angled and isosceles, $BC : BO :: \sqrt{2} : 1$ (188); therefore, *the side of an inscribed square is to radius, as the square root of two is to unity.*

PROBLEM.

271. *To inscribe a regular hexagon and an equilateral triangle in a given circle.*

Fig. 158. *Solution.* Let us suppose the problem resolved, and that AB (*fig.* 158) is a side of the inscribed hexagon; if we draw the radii AO, OB, the triangle AOB will be equilateral.

For the angle AOB is the sixth part of four right angles; thus, if we consider the right angle as unity, we shall have

$$AOB = \tfrac{4}{6} = \tfrac{2}{3}.$$

The two other angles ABO, BAO, of the same triangle, taken together, $= 2 - \frac{2}{3} = \frac{4}{3}$; and, as they are equal to each other, each of them $= \frac{2}{3}$; hence the triangle ABO is equilateral; therefore the side of the inscribed hexagon is equal to radius.

It follows from this, that in order to inscribe a regular hexagon in a given circle, the radius is to be applied six times round on the circumference, which will bring it to the point, from which the operation commenced.

The hexagon $ABCDEF$ being inscribed, if the vertices of the alternate angles A, C, E, be joined, an equilateral triangle ACE will be formed.

272. *Scholium.* The figure $ABCO$ is a parallelogram, and a rhombus, since $AB = BC = CO = AO$; therefore, the sum of the squares of the diagonals being equal to the sum of the squares of the sides (195),

$$\overline{AC}^2 + \overline{BO}^2 = 4\overline{AB}^2 = 4\overline{BO}^2;$$

subtracting $\overline{BO}^2$ from each we shall have

$$\overline{AC}^2 = 3\overline{BO}^2$$

hence $\overline{AC}^2 : \overline{BO}^2 :: 3 : 1$

or $AC : BO :: \sqrt{3} : 1$;

therefore *the side of an inscribed equilateral triangle is to radius as the square root of three is to unity.*

PROBLEM.

273. *To inscribe in a given circle a regular decagon, also a pentagon and a regular polygon of fifteen sides.*

Fig. 159. *Solution.* Divide the radius AO (*fig.* 159) in extreme and mean ratio at the point M (240), take the chord AB equal to the

greater segment OM, and AB will be the side of a regular decagon, which is to be applied ten times round on the circumference.

For, by joining MB, we have, by construction,

$$AO : OM :: OM : AM,$$

or, because $AB = OM$,

$$AO : AB :: AB : AM;$$

therefore the triangles ABO, AMB, having an angle A common and the sides about this angle proportional, are similar (208). The triangle OAB is isosceles; consequently the triangle AMB is also isosceles, and $AB = BM$. Besides, $AB = OM$; hence also $BM = OM$; therefore the triangle BMO is isosceles.

The angle AMB, the exterior angle of the isosceles triangle BMO, is double of the interior angle O (78). Now the angle

$$AMB = MAB;$$

consequently the triangle OAB is such that each of the angles at the base OAB, OBA, is double of the angle at the vertex O, and the three angles of the triangle are equal to five times the angle O, and thus the angle O is a fifth part of two right angles, or the tenth part of four right angles; therefore the arc AB is the tenth part of the circumference, and the chord AB is the side of a regular decagon.

274. *Corollary* I. If the alternate vertices A, C, E, &c. of the decagon be joined, a regular pentagon $ACEGI$ will be formed.

275. *Corollary* II. AB being always the side of a decagon, let AL be the side of a hexagon; then the arc BL will be, with respect to the circumference, $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$; therefore the chord BL will be the side of a regular polygon of 15 sides. It is manifest, at the same time, that the arc CL is a third of CB.

276. *Scholium.* A regular polygon being formed, if the arcs subtended by the sides be bisected and chords to these half arcs be drawn, a regular polygon will be formed of double the number of sides. Thus by means of the square we may inscribe successively regular polygons of 8, 16, 32, &c., sides. Likewise by means of the hexagon we may inscribe regular polygons of 12, 24, 48, &c., sides; with the decagon, polygons of 20, 40, 80, &c., sides; with the regular polygon of fifteen sides, polygons of 30, 60, 120, &c., sides.*

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* It was supposed, for a long time, that these were the only polygons which could be inscribed by the processes of elementary geom-

PROBLEM.

Fig. 160 277. *A regular inscribed polygon* ABCD *&c.* (fig. 160) *being given, to circumscribe about the same circle a similar polygon.*

Solution. At the point T, the middle of the arc AB, draw the tangent GH, which will be parallel to AB (112); do the same with each of the other arcs BC, CD, &c.; these tangents will form, by their intersections, the regular circumscribed polygon $GHIK$ &c., similar to the inscribed polygon.

It will be readily perceived, in the first place, that the three points O, B, H, are in a right line, for the right-angled triangles OTH, OHN, have the common hypothenuse OH, and the side $OT = ON$; they are consequently equal (126), and the angle $TOH = HON$, and the line OH passes through the point B, the middle of the arc TN. For the same reason, the point I is in OC produced, &c. But, since GH is parallel to AB, and HI to BC, the angle $GHI = ABC$ (67); in like manner $HIK = BCD$, &c.; hence the angles of the circumscribed polygon are equal to those of the inscribed polygon. Moreover, on account of these same parallels

$$GH : AB :: OH : OB,$$

and $$HI : BC :: OH : OB,$$

hence $$GH : AB :: HI : BC.$$

But $AB = BC$; consequently $GH = HI$. For the same reason $HI = IK$, &c.; consequently the sides of the circumscribed polygon are equal to each other; therefore this polygon is regular and similar to the inscribed polygon.

278. *Corollary* I. Reciprocally, if the circumscribed polygon $GHIK$ &c., be given, and it is proposed to construct, by means of it, the inscribed polygon $ABCD$ &c., it is evidently sufficient to draw to the vertices G, H, I, &c., of the given polygon the lines OG, OH, OI, &c., which will meet the circumference at the points A, B, C, &c., and then to join these points by the

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etry, or, which amounts to the same thing, by the resolution of equations of the first and second degree. Dut M. Gaus has shown in a work, entitled *Disquisitiones Arithmeticæ*, Lipsiæ, 1801, that we may, by similar methods, inscribe a regular polygon of seventeen sides and in general one of $2^n + 1$ sides, provided that $2^n + 1$ be a prime number.

chords *AB*, *BC*, *CD*, &c., which will form the inscribed polygon. We might also, in this case, simply join the points of contact, *T*, *N*, *P*, &c., by the chords *TN*, *NP*, *PQ*, &c., which would equally form an inscribed polygon similar to the circumscribed one.

279. *Corollary* II. There may be circumscribed, about a given circle, all the regular polygons which can be inscribed within it; and, reciprocally, there may be inscribed, within a circle, all the polygons that can be circumscribed about it.

THEOREM.

280. *The area of a regular polygon is equal to the product of its perimeter by half of the radius of the inscribed circle.*

Demonstration. Let there be, for example, the regular polygon *GHIK* &c. (*fig.* 160); the triangle *GOH*, for example, has for its measure $GH \times \frac{1}{2}OT$, the triangle *OHI* has for its measure $HI \times \frac{1}{2}ON$. But $ON = OT$; consequently the two triangles united have for their measure $(GH + HI) \times \frac{1}{2}OT$. By proceeding thus with the other triangles, it is evident that the sum of all the triangles, or the entire polygon, has for its measure the sum of the bases *GH*, *HI*, *IK*, &c., or the perimeter of the polygon, multiplied by $\frac{1}{2}OT$, half of the radius of the inscribed circle. Fig. 160.

281. *Scholium.* The radius of the inscribed circle is the same as the perpendicular let fall from the centre upon one of the sides.

THEOREM.

282. *The perimeters of regular polygons of the same number of sides are as the radii of the circumscribed circles, and also as the radii of the inscribed circles; and their surfaces are as the squares of these same radii.*

Demonstration. Let *AB* (*fig.* 161) be a side of one of the polygons in question, *O* its centre, and *OA* the radius of the circumscribed circle, and *OD*, perpendicular to *AB*, the radius of the inscribed circle; and let *a b* be the side of another polygon, similar to the former, *o* its centre, *o a* and *o d* the radii of the circumscribed and inscribed circles. Fig. 161.

The perimeters of the two polygons are to each other as the sides *AB*, *a b* (221). Now the angles *A* and *a* are equal, being each half of the angle of the polygon; the same may be said of the angles B and b; therefore the triangles ABO, abo, are similar, as also the right-angled triangles ADO, ado;

hence $AB : ab :: AO : ao :: DO : do$;

consequently the perimeters of the polygons are to each other as the radii AO, ao, of the circumscribed circles, and also as the radii DO, do, of the inscribed circles.

The surfaces of these same polygons are to each other as the squares of the homologous sides AB, ab (221); they are therefore also as the squares of the radii of the circumscribed circles AO, ao, and as the squares of the radii of the inscribed circles DO, do.

LEMMA.

283. *Every curved line, or polygon, which encloses, from one* Fig. 162. *extremity to the other, a convex line* AMB (fig. 162), *is greater than the enclosed line* AMB.

Demonstration. We have already said that, by a convex line, we understand a curved line or polygon, or a line consisting in part of a curve and in part of a polygon, such that a straight line cannot cut it in more than two points (83). If the line AMB had re-entering parts or sinuosities, it would cease to be convex, because, as will be readily perceived, it might be cut by a straight line in more than two points. The arcs of a circle are essentially convex; but the proposition under consideration extends to every line, which fulfils the condition stated.

This being premised, if the line AMB be not smaller than any of those lines which enclose it, there is among these last a line smaller than any of the others, which is less than AMB, or at least equal to AMB. Let $ACDEB$ be this enclosing line; between these two lines draw at pleasure the straight line PQ, which does not meet the line AMB, or at most only touches it; the straight line PQ is less than $PCDEQ$ (3); consequently if, instead of $PCDEQ$, we substitute the straight line PQ, we shall have the enclosing line $APQB$, less than $APDQB$. But, by hypothesis, this must be the shortest of all; this hypothesis then cannot be maintained; therefore each of the enclosing lines is greater than AMB.

284. *Scholium.* After the same manner, it may be demonstrated, without any restriction, that a line which is convex and

returns into itself, *AMB* (*fig.* 163), is less than any line which Fig. 163. encloses it on all sides, whether the enclosing line *FHG* touches *AMB* in one or more points, or whether it surrounds it without touching it.

LEMMA.

285. *Two concentric circles being given, there may always be inscribed, in the greater, a regular polygon, the sides of which shall not meet the circumference of the smaller; and there may also be circumscribed, about the smaller, a regular polygon, the sides of which shall not meet the circumference of the greater; so that on the whole the sides of the polygon described shall be contained between the two circumferences.*

Demonstration. Let *CA*, *CB* (*fig.* 164), be the radii of the two Fig. 164. given circles. At the point *A* draw the tangent *DE* terminating, at the greater circumference, in *D* and *E*. Inscribe, in the greater circumference, one of the regular polygons, which can be inscribed by the preceding problems, and bisect the arcs subtended by the sides, and draw the chords of these half arcs; and a regular polygon will be described of double the number of sides. Continue to bisect the arcs until one is obtained which is smaller than *DBE*. Let *MBN* be this arc, the middle of which is supposed to be in *B*; it is evident that the chord *MN* will be further from the centre than *DE*, and that thus the regular polygon, of which *MN* is a side, cannot meet the circumference, of which *CA* is the radius.

The same things being supposed, join *CM* and *CN*, which meet the tangent *DE* in *P* and *Q*; *PQ* will be the side of a polygon circumscribed about the smaller circumference similar to the polygon inscribed in the greater, the side of which is *MN*. Now it is evident that the circumscribed polygon, which has for its side *PQ*, cannot meet the greater circumference, since *CP* is less than *CM*.

There may, therefore, by the same construction, be a regular polygon inscribed in the greater circumference, and a similar polygon circumscribed about the smaller, which shall have their sides comprehended between the two circumferences.

286. *Scholium.* If we have two concentric sectors *FCG*, *ICH*, we can likewise inscribe, in the greater, *a portion of a regular polygon*, or circumscribe, about the smaller, a portion of a similar

polygon, so that the perimeters of the two polygons would be comprehended between the two circles. It is only necessary to divide the arc *FBG* successively into 2, 4, 8, 16, &c., equal parts, until one is obtained smaller than *DBE.*

By a *portion of a regular polygon*, as the phrase is here used, is to be understood the figure terminated by a series of equal chords, inscribed in the arc *FG*, from one extremity to the other. This portion has the principal properties of a regular polygon, it has its angles equal, and its sides equal; it is, at the same time, capable of being inscribed in, and circumscribed about a cirle; it does not, however, make a part of a regular polygon, properly so called, except when the arc, subtended by one of these sides, is an aliquot part of the circumference.

THEOREM.

287. *The circumferences of circles are as their radii, and their surfaces are as the squares of their radii.*

Fig. 165. *Demonstration.* Denoting, by *circ. CA* and *circ. OB* (*fig.* 165), the circumferences of the circles whose radii are *CA* and *OB*, we say that *circ. CA* : *circ. OB* : : *CA* : *OB.*

For, if this proportion be not true, *CA* will be to *OB* as *circ. CA* is to a fourth term either greater or less than *circ. OB.* Let us suppose that it is less, and that, if possible,

CA : *OB* : : *circ. CA* : *circ. OD.*

Inscribe, in the circumference of which *OB* is the radius, a regular polygon *EFGKLE*, whose sides shall not meet the circumference of the circle, whose radius is *OD* (285); inscribe a similar polygon *MNPSTM* in the circle whose radius is *CA.*

This being done, since the polygons are similar, their perimeters *MNPSM*, *EFGKE*, are to each other as the radii *CA*, *OB*, of the circumscribed circles (282), and we have

MNPSM : *EFGKE* : : *CA* : *OB*;

but, by hypothesis,

CA : *OB* : : *circ. CA* : *circ. OD*;

therefore *MNPSM* : *EFGKE* : : *circ. CA* : *circ. OD.* Now this proportion is impossible, because the perimeter *MNPSM* is less than *circ. CA* (283), while *EFGKE* is greater than the *circ. OD*; therefore it is impossible that *CA* should be to *OB* as *circ. CA* is to a circumference less than *circ. OB*; or, in other words, it is impossible that the radius of one circle should be to that of another as the circumference of the first is to a circumference less than that of the second.

It follows, moreover, from what has been said, that CA cannot be to OB as *circ.* CA is to a circumference greater than *circ.* OB; for, if this were the case, we should have by *inversion*,

$OB : CA$: : a circumference greater than *circ.* OB : *circ.* CA,
or, which is the same thing,

$OB : CA$ :: *circ.* OB : a circumference less than *circ.* CA;
therefore the radius of one circle may be to the radius of another, as the circumference described upon the former is to a circumference less than the one described upon the latter, which has been shown to be impossible.

Since the fourth term of the proportion $CA : OB$ :: *circ.* $CA : X$ can be neither less nor greater than *circ.* OB, it must be equal to *circ.* OB; therefore the circumferences of circles are as their radii.

By a construction and course of reasoning entirely similar, it may be demonstrated that the surfaces of circles are as the squares of their radii.

We shall not enter into further details upon this proposition, which is indeed a corollary from the next.

288. *Corollary.* Similar arcs AB, DE (*fig.* 166), are as their radii AC, DO; and similar sectors ACB, DOE, are as the squares of their radii. Fig. 166.

For, since the arcs are similar, the angle C is equal to the angle O (163); now the angle C is to four right angles as the arc AB is to the entire circumference described upon the radius AC (122), and the angle O is to four right angles as the arc DE is to the circumference described upon the radius OD; therefore the arcs AB, DE, are to each other as the circumferences of which they are respectively a part; and these circumferences are as the radii AC, DO; therefore

$$\text{arc } AB : \text{arc } DE :: AC : DO.$$

For the same reason the sectors ACB, DOE, are as the entire circles; but the entire circles are as the squares of their radii; therefore

$$\text{sect. } ACB : \text{sect. } DOE :: \overline{AC}^2 : \overline{DO}^2.$$

THEOREM.

289. *The area of a circle is equal to the product of its circumference by half of the radius.*

Demonstration. Denoting by *surf.* CA the surface or area of a circle whose radius is CA, we say that

$$surf.\ CA = \tfrac{1}{2}\ CA \times circ.\ CA.$$

Fig. 167. If $\frac{1}{2}\ CA \times circ.\ CA$ (*fig.* 167) be not the area of the circle of which CA is the radius, this quantity will be the measure of a circle either greater or less. Let us suppose, in the first place, that it is the measure of a greater circle, and that, if it be possible, $\frac{1}{2}\ CA \times circ.\ CA = surf.\ CB$.

About the circle, of which CA is the radius, circumscribe a regular polygon $DEFG$ &c., the sides of which shall not meet the circumference of the circle whose radius is CB (285); the surface of this polygon will be equal to its perimeter

$$DE + EF + FG + \&c.,$$

multiplied by $\frac{1}{2}\ AC$ (280). But the perimeter of the polygon is greater than that of the inscribed circle, since it encloses it on all sides; consequently the surface of the polygon $DEFG$ &c. is greater than $\frac{1}{2}\ AC \times circ.\ AC$, which, by hypothesis, is the measure of the circle, of which CB is the radius; hence the polygon would be greater than the circle; but it is less, since it is contained within it; therefore it is impossible that

$$\tfrac{1}{2}\ CA \times circ.\ CA$$

should be greater than *surf.* CA, or, in other words, it is impossible that the circumference of a circle multiplied by half of the radius should be the measure of a greater circle.

Again, this same product cannot be the measure of a less circle; and, not to change the figure, I will suppose that the circle in question is that whose radius is CB; it is to be proved then, that $\frac{1}{2}\ CB \times circ.\ CB$ cannot be the measure of a less circle, of the circle, for example, whose radius is CA. Let us suppose, if it be possible, that $\frac{1}{2}\ CB \times circ.\ CB = surf.\ CA$.

The same construction being supposed as above, the surface of the polygon $DEFG$ &c. will have for its measure

$$(DE + EF + FG + \&c.) \times \tfrac{1}{2}\ CA;$$

but the perimeter $DE + EF + FG + \&c.$, is less than *circ.* CB which encloses it on all sides; hence the area of the polygon is less than $\frac{1}{2}\ CA \times circ.\ CB$, and for a still stronger reason, less than $\frac{1}{2}$ *CB* × *circ. CB.* This last quantity is, by hypothesis, the measure of the circle of which *CA* is the radius; consequently the polygon would be less than the inscribed circle, which is absurd; it is impossible then that the circumference of a circle multiplied by half of the radius should be the measure of a less circle.

Therefore the circumference of a circle multiplied by half of the radius is the measure of this circle.

290. *Corollary* I. The surface of a sector is equal to the arc of this sector multiplied by half of the radius.

For the sector *ACB* (*fig.* 168) is to the entire circle, as the arc *AMB* is to the entire circumference *ABD* (125), or as *AMB* × $\frac{1}{2}$*AC* is to *ABD* × $\frac{1}{2}$*AC*. But the entire circle is equal to *ABD* × $\frac{1}{2}$*AC*; therefore the sector *ACB* has for its measure *AMB* × $\frac{1}{2}$*AC*. Fig. 168.

291. *Corollary* II. Since the circumferences of circles are as their radii, or as their diameters, calling π the circumference of a circle whose diameter is one, we have this proportion; the diameter of a circle 1 is to its circumference π, as the diameter 2*CA* is to the circumference of a circle whose radius is *CA*,

or $$1 : \pi :: 2CA : circ.\ CA;$$

hence $$circ.\ CA = 2\pi \times CA.$$

Multiplying each member by $\frac{1}{2}$ *CA*, we have

$$\tfrac{1}{2}\,CA \times circ.\ CA = \pi \times \overline{CA}^2,$$

or $$surf.\ CA = \pi \times \overline{CA}^2;$$

therefore, *the surface of a circle is equal to the product of the square of the radius by the constant number* π, *which represents the circumference of a circle whose diameter is* 1, *or the ratio of the circumference to the diameter.*

In like manner, the surface of a circle whose radius is *OB*, is equal to $\pi \times \overline{OB}^2$. But

$$\pi \times \overline{CA}^2 : \pi \times \overline{OB}^2 :: \overline{CA}^2 : \overline{OB}^2;$$

therefore, *the surfaces of circles are to each other as the squares of their radii*, which agrees with the preceding theorem.

292. *Scholium.* We have already said, that the problem of the quadrature of the circle consists in finding a square equal in surface to a circle whose radius is known; now we have just shown that a circle is equivalent to a rectangle contained by the

circumference and half of the radius, and this rectangle is changed into a square by finding a mean proportional between its two dimensions (243). Thus the problem of the quadrature of the circle reduces itself to finding the circumference, when the radius is known; and, for this purpose, it is sufficient to know the ratio of the circumference to the radius or to the diameter.

Hitherto we have not been able to obtain this ratio except by approximation; but the process has been carried so far, that a knowledge of the exact ratio would have no real advantage over the approximate ratio. Indeed this question, which occupied much of the attention of geometers, when the methods of approximation were less known, is now ranked among those idle questions which engage the attention of such only as have scarcely attained to the first principles of geometry.

Archimedes proved that the ratio of the circumference to the diameter is comprehended between $3\frac{10}{70}$ and $3\frac{10}{71}$; thus $3\frac{1}{7}$ or $\frac{22}{7}$ is a value already approaching very near to the number, which we have represented by π; and this first approximation is much in use on account of its simplicity. *Metius* gave a much nearer value of this ratio in the expression $\frac{355}{113}$. Other calculators have found the value of π, developed to a certain number of decimals, to be 3,1415926535897932 &c., and some have had the patience to extend these decimals to the hundred and twenty-seventh, and even to the hundred and fortieth. Such an approximation may evidently be taken as equivalent to the truth, and the roots of imperfect powers are not better known.

We shall explain, in the following problems, two elementary methods, the most simple, for obtaining these approximations.

PROBLEM.

293. *The surface of a regular inscribed polygon and that of a similar circumscribed polygon being given, to find the surfaces of regular inscribed and circumscribed polygons of double the number of sides.*

Fig. 169. *Solution.* Let AB (*fig.* 169) be the side of a given inscribed polygon, EF parallel to AB, that of a similar circumscribed polygon, C the centre of the circle; if we draw the chord AM, and the tangents AP, BQ, the chord AM will be the side of an inscribed polygon of double the number of sides, and PQ double

of PM will be that of a similar circumscribed polygon (277); and, as the different angles of the polygon equal to ACM will admit of the same construction, it is sufficient to consider the angle ACM only, and the triangles here contained will be to each other as the entire polygons. Let A be the surface of the inscribed polygon whose side is AB, B the surface of a similar circumscribed polygon, A' the surface of a polygon whose side is AM, B' the surface of a similar circumscribed polygon. A and B are known, and it is proposed to find A' and B'.

1. The triangles ACD, ACM, the common vertex of which is A, are to each other as their bases CD, CM; moreover, these triangles are as the polygons A and A', of which they are respectively a part; hence

$$A : A' :: CD : CM.$$

The triangles CAM, CME, the common vertex of which is M, are to each other as their bases CA, CE; these same triangles are also as the polygons A' and B, of which they are respectively a part; hence

$$A' : B :: CA : CE.$$

But, on account of the parallels AD, ME,

$$CD : CM :: CA : CE;$$

therefore $$A : A' :: A' : B;$$

that is, the polygon A', one of those which is sought, is a mean proportional between the two known polygons A and B; consequently $$A' = \sqrt{A \times B}.$$

2. On account of the common altitude CM, the triangle CPM is to the triangle CPE as PM is to PE; but, as the line CP bisects the angle MCE (201),

$$PM : PE :: CM : CE :: CD : CA \text{ or } CM :: A : A';$$

hence $$CPM : CPE :: A : A',$$

and $$CPM : CPM + CPE \text{ or } CME :: A : A + A';$$

also $$2CPM \text{ or } CMPA : CME :: 2A : A + A'.$$

But $CMPA$ and CME are to each other as the polygons B' and B, of which they are respectively a part; we have then

$$B' : B :: 2A : A + A'.$$

Now A' has already been determined; and this new proportion will give the determination of B', namely,

$$B' = \frac{2A \times B}{A + A'};$$

therefore, by means of the polygons A and B, it is easy to find the polygons A' and B', which have double the number of sides.

PROBLEM.

294. *To find the approximate ratio of the circumference of a circle to its diameter.*

Solution. Let the radius of the circle be $= 1$, the side of the inscribed square will be $\sqrt{2}$ (270), that of the circumscribed square will be equal to the diameter 2; hence the surface of the inscribed square $= 2$, and that of the circumscribed square $= 4$. Now, if we make $A = 2$, and $B = 4$, we shall find, by the preceding problem, the inscribed octagon $A' = \sqrt{8} = 2,8284271$, and the circumscribed octagon $B' = \frac{16}{2 + \sqrt{8}} = 3,3137085$. Knowing thus the inscribed and circumscribed octagons, we can find, by means of them, the polygons of double the number of sides; we now suppose $A = 2,8284271$, $B = 3,3137085$, and we shall have $A' = \sqrt{A \times B} = 3,0614674$, and $B' = \frac{2A \times B}{A + A'} = 3,1825979$. These polygons of 16 sides will serve to find those of 32 sides, and we may proceed in this manner, till there is no difference between the inscribed and circumscribed polygons, at least for the number of decimals to which the calculation is carried, which in this example is seven. Having arrived at this point, we conclude that the circle is equal to the last result, for the circle must always be comprehended between the inscribed and circumscribed polygons; therefore, if these do not differ from each other for a certain number of decimals, the circle will not differ from them for the same number.

See the calculation of these polygons continued till they give the same result for the seven first decimals.

Number of sides.	Inscribed polygon.	Circumscribed polygon.
4	2,0000000	4,0000000
8	2,8284271	3,3137085
16	3,0614674	3,1825979
32	3,1214451	3,1517249
64	3,1365485	3,1441148
128	3,1403311	3,1422236
256	3,1412772	3,1417504
512	3,1415138	3,1416321
1024	3,1415729	3,1416025
2048	3,1415877	3,1415951
4096	3,1415914	3,1415933
8192	3,1415923	3,1415928
16384	3,1415925	3,1415927
32768	3,1415926	3,1415926

Hence we conclude that the surface of the circle = 3,1415926.

There might be some doubt with respect to the last decimal, on account of the error arising from the parts neglected; but we have extended the calculation to one decimal more in order to be assured of the correctness of the above result to the last figure.

Since the surface of a circle is equal to the product of the semicircumference by the radius, the radius being 1, the semicircumference will be 3,1415926; or, the diameter being 1, the circumference will be 3,1415926; therefore the ratio of the circumference to the diameter, above denoted by π, is equal to 3,1415926.

LEMMA.

295. *The triangle* CAB (fig. 170) *is equivalent to the isosceles triangle* DCE, *which has the same angle* C, *and of which the side* CE *equal to* CD *is a mean proportional between* CA *and* CB. *Moreover, if the angle* CAB *is a right angle, the perpendicular* CF *let fall upon the base of the isosceles triangle will be a mean proportional between the side* CA *and the half sum of the sides* CA, CB. Fig. 170.

Demonstration. 1. On account of the common angle C, the triangle ABC is to the isosceles triangle DCE as $AC \times CB$ is to $DC \times CE$ or $\overline{DC}^2$ (216); consequently these triangles are equivalent, when $\overline{DC}^2 = AC \times CB$, or when DC is a mean proportional between AC and CB.

2. As the perpendicular CGF bisects the angle ACB,

$$AG : GB :: AC : CB \quad (201),$$

whence, by *composition*,

$$AG : AG + GB \text{ or } AB :: AC : AC + CB;$$

but $AG : AB ::$ triangle ACG : triangle ACB or $2CDF$; moreover, if the angle A is a right angle, the right-angled triangles ACG, CDF, are similar; whence

$$ACG : CDF :: \overline{AC}^2 : \overline{CF}^2;$$

or

$$ACG : 2CDF :: \overline{AC}^2 : 2\overline{CF}^2;$$

therefore

$$\overline{AC}^2 : 2\overline{CF}^2 :: AC : AC + CB.$$

If we multiply the two terms of the second ratio by AC, the antecedents will become equal, and we shall consequently have

$$2\overline{CF}^2 = AC \times (AC + CB), \text{ or } \overline{CF}^2 = AC \times \left(\frac{AC + CB}{2}\right);$$

therefore, if the angle *A* is a right angle, the perpendicular *CF* is a mean proportional between the side *AC* and half the sum of the sides *AC*, *CB*.

PROBLEM.

296. *To find a circle which shall differ as little as we please from a given regular polygon.*

Solution. Let there be given, for example, the square *BMNP* (*fig.* 171); let fall from the centre *C* the perpendicular *CA* upon the side *MB*, and join *CB*. Fig. 171.

The circle described upon the radius *CA* is inscribed in the square, and the circle described upon the radius *CB* is circumscribed about this square; the first will be less than the square, and the second will be greater; it is proposed to reduce these limits.

Take *CD* and *CE* each equal to a mean proportional between *CA* and *CB*, and join *ED*; the isosceles triangle *CDE* will be equivalent to the triangle *CAB* (295); let the same be done with respect to each of the eight triangles which compose the square, and there will be formed a regular octagon equivalent to the square *BMNP*. The circle described upon *CF*, a mean proportional between *CA* and $\frac{CA + CB}{2}$, will be inscribed in the octagon, and the circle described upon *CD*, as a radius, will be circumscribed about it. Thus the first will be less and the second greater than the given square.

If we change, in the same manner, the right-angled triangle *CDF* into an equivalent isosceles triangle, we shall form in this way a regular polygon of sixteen sides equivalent to the proposed square. The circle inscribed in this polygon will be less than the square, and the circle circumscribed about it will be greater.

We can proceed in this manner till the ratio between the radius of the inscribed circle and that of the circumscribed circle shall differ as little as we please from equality. Then either of these circles may be regarded as equivalent to the proposed square.

297. *Scholium.* To exhibit the result of this investigation of the successive radii, let a be the radius of the circle inscribed in one of the polygons, and b the radius of the circle circumscribed about the same polygon; and let a', b', be similar radii to the

next polygon of double the number of sides. According to what has been demonstrated, b' is a mean proportional between a and b, and a' is a mean proportional between a and $\frac{a+b}{2}$; so that we have

$$b' = \sqrt{a \times b}, \text{ and } a' = \sqrt{a \times \frac{a+b}{2}};$$

hence the radii a and b of one polygon being known, the radii a', b', of the following polygon are easily deduced; and we may proceed in this manner till the difference between the two radii shall become insensible; then either of these radii may be taken for the radius of a circle equivalent to the proposed square or polygon.

This method may be readily applied to lines, since it consists in finding successive mean proportionals between known lines; but it succeeds still better by means of numbers, and it is one of the most convenient, that elementary geometry can furnish, for finding expeditiously the approximate ratio of the circumference of a circle to its diameter. Let the side of the square be equal to 2, the first inscribed radius *CA* will be 1, and the first circumscribed radius *CB* will be $\sqrt{2}$ or 1,4142136. Putting then $a = 1$, and $b = 1{,}4142136$, we shall have

$$b' = \sqrt{a \times b} = \sqrt{1 \times 1{,}4142136} = 1{,}1892071;$$

$$a' = \sqrt{a \times \frac{a+b}{2}} = \sqrt{1 \times \frac{1 + 1{,}4142136}{2}} = 1{,}0986841.$$

These numbers may be used in calculating the succeeding ones according to the law of continuation.

See the result of this calculation extended to seven or eight figures by means of a table of common logarithms.

Radii of the circumscribed circles.	Radii of the inscribed circles.
1,4142136	1,0000000.
1,1892071	1,0986841.
1,1430500	1,1210863.
1,1320149	1,1265639.
1,1292862	1,1279257.
1,1286063	1,1282657.

The first half of the figures being now the same in both, we can take the arithmetical instead of the geometrical means, since they do not differ from each other except in the remoter decimals (*Alg.* 102). The operation is thus greatly abridged, and the results are,

1,1284360	1,1283508.
1,1283934	1,1283721.
1,1283827	1,1283774.
1,1283801	1,1283787.
1,1283794	1,1283791.
1,1283792	1,1283792.

Hence 1,1283792 is very nearly the radius of a circle equal in surface to a square whose side is 2. From this it is easy to find the ratio of the circumference of a circle to its diameter; for it has been demonstrated that the surface of a circle is equal to the square of the radius multiplied by the number π; therefore if we divide the surface 4 by the square of 1,1283792, we shall have the value of π equal to 3,1415926 &c., as determined by the other method.

Appendix to the Fourth Section.

DEFINITIONS.

298. Among quantities of the same kind that which is greatest is called a *maximum;* and that which is smallest a *minimum.*

Thus the diameter of a circle is a *maximum* among all the straight lines drawn from one point of the circumference to another, and a perpendicular is a *minimum* among all the straight lines drawn from a given point to a given straight line.

299. Those figures which have equal perimeters are called *isoperimetrical figures.*

THEOREM.

300. *Among triangles of the same base and the same perimeter that is a* maximum *in which the two undetermined sides are equal.*

Fig. 172. *Demonstration.* Let $AC = CB$ (*fig.* 172), and

$$AM + MB = AC + CB;$$

the isosceles triangle ACB will be greater than the triangle AMB of the same base and the same perimeter.

From the point C, as a centre, and with the radius $CA = CB$, describe a circle meeting CA produced in D; join DB; and the angle DBA, inscribed in a semicircle is a right angle (128). Produce the perpendicular DB towards N, and make $MN = MB$, and join AN. From the points M and C let fall upon DN the perpendiculars MP and CG. Since $CB = CD$, and $MN = MB$, $AC + CB = AD$, and $AM + MB = AM + MN$. But

$$AC + CB = AM + MB;$$

consequently $AD = AM + MN$; therefore $AD > AN$.

Now, if the oblique line AD is greater than the oblique line AN, it must be at a greater distance from the perpendicular AB (52); hence $DB > BN$, and BG the half DB is greater than BP the half BN. But the triangles ABC, ABM, which have the same base AB, are to each other as their altitudes BG, BP; therefore, since $BG > BP$, the isosceles triangle ABC is greater than the triangle ABM of the same base and same perimeter which is not isosceles.

THEOREM.

301. *Among polygons of the same perimeter and of the same number of sides, that is a* maximum *which has its sides equal.*

Demonstration. Let $ABCDEF$ (*fig.* 173) be the *maximum* polygon; if the side BC is not equal to CD, make, upon the base BD, an isosceles triangle BOD, having the same perimeter as BCD, the triangle BOD will be greater than BCD (300), and consequently the polygon $ABODEF$ will be greater than $ABCDEF$; this last then will not be a *maximum* among all those of the same perimeter and the same number of sides, which is contrary to the supposition. Hence BC must be equal to CD; and, for the same reason, we shall have $CD = DE$, $DE = EF$, &c.; therefore all the sides of the *maximum* polygon are equal to each other. Fig. 173.

THEOREM.

302. *Of all triangles formed with two given sides making any angle at pleasure with each other, the* maximum *is that in which the two given sides make a right angle.*

Demonstration. Let there be the two triangles BAC, BAD (*fig.* 174), which have the side AB common, and the side $AC = AD$; if the angle BAC is a right angle, the triangle BAC will be greater than the triangle BAD, in which the angle A is acute or obtuse. Fig. 174.

For, the base AB being the same, the two triangles BAC, BAD, are as their altitudes AC, DE. But the perpendicular

DE is less than the oblique line *AD* or its equal *AC*; therefore the triangle *BAD* is less than *BAC*.

THEOREM.

303. *Of all polygons formed of given sides and one side to be taken of any magnitude at pleasure, the* maximum *must be such that all the angles may be inscribed in a semicircle of which the unknown side shall be the diameter.*

Fig. 175. *Demonstration.* Let *ABCDEF* (*fig.* 175), be the greatest of the polygons formed of the given sides *AB*, *BC*, *CD*, *DE*, *EF*, and the side *AF* taken at pleasure; draw the diagonals *AD*, *DF*. If the angle *ADF* is not a right angle, we can, by preserving the parts *ABCD*, *DEF*, as they are, augment the triangle *ADF*, and consequently the entire polygon by making the angle *ADF* a right angle, according to the preceding proposition; but this polygon can no longer be augmented, since it is supposed to have attained its *maximum*; therefore the angle *ADF* is already a right angle. The same may be said of the angles *ABF*, *ACF*, *AEF*; hence all the angles *A*, *B*, *C*, *D*, *E*, *F*, of the *maximum* polygon are inscribed in a semicircle of which the undetermined side *AF* is the diameter.

304. *Scholium.* This proposition gives rise to a question, namely, whether there are several ways of forming a polygon with given sides and one unknown side, the unknown side being the diameter of the semicircle in which the other sides are inscribed. Before deciding this question it is proper to observe that, if the same chord *AB* subtends arcs described upon different radii *AC*, *AD* (*fig.* 176), the angle at the centre subtended by this chord will be least in the circle of the greatest radius; thus $ACB < ADB$. For $ADO = ACD + CAD$ (78); therefore $ACD < ADO$, and, each being doubled, we have $ACB < ADB$. Fig. 176.

THEOREM.

305. *There is but one way of forming a polygon* ABCDEF, Fig. 175. (fig. 175) *with given sides and one side unknown, the unknown side being the diameter of the semicircle in which the others are inscribed.*

Demonstration. Let us suppose that we have found a circle which satisfies the question; if we take a greater circle, the chords *AB*, *BC*, *CD*, &c., answer to angles at the centre that are smaller. The sum of the angles at the centre will accordingly be less than two right angles; thus the extremities of the given sides will not terminate in the extremities of a diameter. The contrary will occur if we take a smaller circle; therefore the polygon under consideration can be inscribed in only one circle.

306. *Scholium.* We can change at pleasure the order of the sides *AB*, *BC*, *CD*, &c., and the diameter of the circumscribed circle will always be the same as well as the surface of the polygon; for, whatever be the order of the arcs *AB*, *BC*, *CD*, &c., it is sufficient that their sum makes a semicircumference, and the polygon will always have the same surface, since it will be equal to the semicircle minus the segments *AB*, *BC*, *CD*, &c., the sum of which is always the same.

THEOREM.

307. *Of all polygons formed of given sides the* maximum *is that which can be inscribed in a circle.*

Demonstration. Let *ABCDEFG* (*fig.* 177) be an inscribed polygon, and $abcdefg$ one that does not admit of being inscribed, having its corresponding sides equal to those of the former, namely, $ab = AB$, $bc = BC$, $cd = CD$, &c.; the inscribed polygon will be greater than the other. Fig. 177.

Draw the diameter *EM*, and join *AM*, *MB*; upon $ab = AB$ construct the triangle abm equal to *ABM*, and join em.

According to what has just been demonstrated (303), the polygon *EFGAM* is greater than $efgam$, unless this last can also be inscribed in a semicircle having em for its diameter, in which case the two polygons would be equal (305). For the same reason the polygon *EDCBM* is greater than $edcbm$, with the exception of the case in which they are equal. Hence the entire polygon *EFGAMBCDE* is greater than $efgambcde$, unless they should be in all respects equal; but they are not so (161), since one is inscribed in a circle, and the other does not admit of being inscribed; therefore the inscribed polygon is greater than the other. Taking from them respectively the equal triangles *ABM*, abm, we have the inscribed polygon *ABCDEFG* greater than the polygon not inscribed $abcdefg$.

308. *Scholium.* It may be shown, as in art. 305, that there is only one circle and consequently only one *maximum* polygon

which satisfies the question; and this polygon will still have the same surface, whatever change be made in the order of the sides.

THEOREM.

309. *Among polygons of the same perimeter and the same number of sides the regular polygon is a* maximum.

Demonstration. According to art. 301, the *maximum* polygon has all its sides equal; and, according to the preceding theorem, it is such that it may be inscribed in a circle; therefore it is a regular polygon.

LEMMA.

Fig. 178 310. *Two angles at the centre, measured in two different circles, are to each other as the contained arcs divided by their radii; that is, the angle* C : *angle* O :: *the ratio* $\frac{AB}{AC} : \frac{DE}{DO}$ (fig. 178).

Demonstration. With the radius OF equal to AC, describe the arc FG comprehended between the sides OD, OE, produced; on account of the equal radii AC, OF,

$$C : O :: AB : FG \ (122), \text{ or } :: \frac{AB}{AC} : \frac{FG}{FO}.$$

But, on account of the similar arcs FG, DE,

$$FG : DE :: FO : DO \ (288);$$

hence the ratio $\frac{FG}{FO}$ is equal to the ratio $\frac{DE}{DO}$; therefore

$$C : O :: \frac{AB}{AC} : \frac{DE}{DO}.$$

THEOREM.

311. *Of two regular isoperimetrical polygons that is the greater which has the greater number of sides.*

Fig. 179. *Demonstration.* Let DE (*fig.* 179), be half of a side of one of these polygons, O its centre, OE a perpendicular let fall from the centre upon one of the sides†; let AB be half of a side of the other polygon, C its centre, CB a perpendicular to the side let fall from the centre. We suppose the centres O and C to be

situated at any distance OC, and the perpendiculars OE, CB, in the direction OC; thus DOE and ACB will be the semiangles at the centre of the polygons respectively, and as these angles are not equal, the lines CA, OD, being produced, will meet in some point F; from this point let fall upon OC the perpendicular FG; from the points O and C, as centres, describe the arcs GI, GH, terminating in the sides OF, CF.

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† This perpendicular is called in the original *apothême.* No English word has been adopted answering to it.

This being done, we have, by the preceding lemma,

$$O : C :: \frac{GI}{OG} : \frac{GH}{CG};$$

but DE : perimeter of the first polygon :: O : four right angles, and AB : perimeter of the second polygon :: C : four right angles; hence, the perimeters of the polygon being equal,

$$DE : AB :: O : C$$

or $$DE : AB :: \frac{GI}{OG} : \frac{GH}{CG}.$$

Multiplying the antecedents by OG and the consequents by CG, we have $DE \times OG : AB \times CG :: GI : GH.$

But the similar triangles ODE, OFG, give

$$OE : OG :: DE : FG,$$

whence $DE \times OG = OE \times FG$;

in like manner $AB \times CG = CB \times FG$;

consequently $OE \times FG : CB \times FG :: GI : GH.$

or $OE : CB :: GI : GH.$

If therefore it is made evident that the arc GI is greater than the arc GH, it will follow that the perpendicular OE is greater than CB.

On the other side of CF let there be constructed the figure CKx equal to CGx, so that we may have $CK = CG$, the angle $HCK = HCG$, and the arc $Kx = xG$; the curve KxG enclosing the arc KHG will be greater than this arc (283). Hence Gx half of the curve is greater than GH half of the arc; therefore, for a still stronger reason, GI is greater than GH.

It follows from this that the perpendicular OE is greater than CB; but the two polygons having the same perimeter are to each other as these perpendiculars (280); therefore the polygon, which has for its half side DE, is greater than that which has for its half side AB. The first has the greater number of sides since its angle at the centre is less; therefore of two regular isoperimetrical polygons, that is the greater which has the greater number of sides.

THEOREM.

312. *The circle is greater than any polygon of the same perimeter.*

Demonstration. It has already been proved that among polygons of the same perimeter and the same number of sides, the regular polygon is the greatest; the inquiry is thus reduced to comparing the circle with regular polygons of the same perimeter. Let AI (*fig.* 180) be the half side of any regular polygon, and C its centre. Let there be, in the circle of the same perimeter, the angle $DOE = ACI$, and consequently the arc DE equal to the half side AI;

Fig. 180.

the polygon P : circle C :: triangle ACI : sector ODE,

hence $P : C :: \frac{1}{2} AI \times CI : \frac{1}{2} DE \times OE :: CI : OE.$

Let there be drawn to the point E the tangent EG meeting OD produced in G; the similar triangles ACI, GOE, give the proportion

$$CI : OE :: AI \text{ or } DE : GE;$$

therefore

$$P : C :: DE : GE :: DE \times \tfrac{1}{2} OE : GE \times \tfrac{1}{2} OE,$$

that is, $P : C$ :: sector DOE : triangle GOE;

but the sector is less than the triangle; consequently P is less than C; therefore the circle is greater than any polygon of the same perimeter.

PART SECOND.

SECTION FIRST.

Of Planes and Solid Angles.

DEFINITIONS.

313. A STRAIGHT line is *perpendicular to a plane*, when it is perpendicular to every straight line in the plane which passes through the foot of the perpendicular (326). Reciprocally, the plane, in this case, is perpendicular to the line.

The *foot* of the perpendicular is the point in which the perpendicular meets the plane.

314. A line is *parallel to a plane* when, each being produced ever so far, they do not meet. Also the plane, in this case, is parallel to the line.

315. Two *planes* are *parallel* when, being produced ever so far, they do not meet.

316. It will be demonstrated art. 324, that the common intersection of two planes, which meet each other, is a straight line. This being premised, the *angle* or the mutual *inclination of two planes* is the quantity, whether greater or less, by which they depart from each other; this quantity is measured by the angle contained by two straight lines drawn from the same point perpendicularly to the common intersection, the one being in one of the planes and the other in the other.

This angle may be acute, right, or obtuse.

317. If it is right, the two *planes* are *perpendicular* to each other.

318. A *solid angle* is the angular space comprehended between several planes which meet in the same point.

Thus the solid angle S (*fig.* 199) is formed by the meeting of the planes ASB, BSC, CSD, DSA. Fig. 199.

It requires at least three planes to form a solid angle.

THEOREM.

319. *One part of a straight line cannot be in a plane and another part without it.*

Demonstration. By the definition of a plane (6) a straight line, which has two points in common with the plane, lies wholly in that plane.

320. *Scholium.* In order to determine whether a surface is plane, it is necessary to apply a straight line in different directions to this surface and see if it touches the surface in its whole extent.

THEOREM.

321. *Two straight lines which cut each other are in the same plane, and determine its position.*

Fig. 181. *Demonstration.* Let *AB*, *AC* (*fig.* 181), be two straight lines which cut each other in *A*. Conceive a plane to pass through *AB*, and to be turned about *AB*, until it passes through the point *C*; then, two points *A* and *C* being in the plane, the whole line *AC* is in this plane; therefore the position of the plane is determined by the condition of its containing the two lines *AB*, *AC*.

322. *Corollary* I. A triangle *ABC*, or three points *A*, *B*, *C*, not in the same straight line determine the position of a plane.

Fig. 182. 323. *Corollary* II. Also two parallels *AB*, *CD* (*fig.* 182), determine the position of a plane; for, if the line *EF* be drawn, the plane of the two straight lines *AE*, *EF*, will be that of the parallels *AB*, *CD*.

THEOREM.

324. *If two planes cut each other, their common intersection is a straight line.*

Demonstration. If among the points common to the two planes there were three not in the same straight line, the two planes in question passing each through these three points would make only one and the same plane, which is contrary to the supposition.

THEOREM.

Fig. 183. 325. *If a straight line* AP (fig. 183) *is perpendicular to two others* PB, PC, *which intersect each other at its foot in the plane* MN, *it* *will be perpendicular to every other straight line* PQ *drawn through its foot in the same plane, and thus it will be perpendicular to the plane* MN.

Demonstration. Through a point Q, taken at pleasure in PQ, draw the straight line BC in the angle BPC making $BQ = QC$ (242); join AB, AQ, AC.

The base BC being bisected at the point Q, the triangle BPC will give

$$\overline{PC}^2 + \overline{PB}^2 = 2\overline{PQ}^2 + 2\overline{QC}^2 \quad (194).$$

The triangle BAC will give, in like manner,

$$\overline{AC}^2 + \overline{AB}^2 = 2\overline{AQ}^2 + 2\overline{QC}^2.$$

If we subtract the first equation from the second, and recollect that the triangles APC, APB, each right-angled at P, give $\overline{AC}^2 - \overline{PC}^2 = \overline{AP}^2$, $\overline{AB}^2 - \overline{PB}^2 = \overline{AP}^2$; we shall have

$$\overline{AP}^2 + \overline{AP}^2 = 2\overline{AQ}^2 - 2\overline{PQ}^2;$$

or, by taking half of each member,

$$\overline{AP}^2 = \overline{AQ}^2 - \overline{PQ}^2;$$

hence

$$\overline{AP}^2 + \overline{PQ}^2 = \overline{AQ}^2;$$

therefore the triangle APQ is right-angled at P (193), and AP is perpendicular to PQ.

326. *Scholium.* It is evident, then, not only that a straight line may be perpendicular to all those which pass through its foot in the plane, but that this happens, whenever the line in question is perpendicular to two straight lines drawn in the plane; hence the propriety of the definition art. 313.

327. *Corollary* I. The perpendicular AP is less than any oblique line AQ; therefore it measures the true distance of a point A from the plane PQ.

328. *Corollary* II. Through any given point P in a plane only one perpendicular can be drawn to this plane; for, if there could be two, a plane being supposed to pass through them intersecting the plane MN in PQ, the two perpendiculars would be perpendicular to the line PQ at the same point, and in the same plane, which is impossible (50).

It is also impossible to let fall from a given point, without a plane two perpendiculars to this plane; for let AP, AQ, be these two perpendiculars, then the triangle APQ would have two right angles APQ, AQP, which is impossible.

THEOREM.

329. ***Oblique lines equally distant from the perpendicular are equal; and of two oblique lines unequally distant from the perpendicular, that which is at the greater distance is the greater.***

Fig. 184. *Demonstration.* The angles *APB*, *APC*, *APD* (*fig.* 184), being right angles, if we suppose the distances *PB*, *PC*, *PD*, equal to each other, the triangles *APB*, *APC*, *APD*, have two sides and the included angle respectively equal, they are consequently equal; therefore the hypothenuses or the oblique lines *AB*, *AC*, *AD*, are equal to each other. Likewise if the distance *PE* is greater than *PD* or its equal *PB*, it is evident that the oblique line *AE* will be greater than *AB* or its equal *AD*.

330. *Corollary.* All the equal oblique lines *AB*, *AC*, *AD*, &c., terminate in the circumference of a circle *BCD* described about the foot of the perpendicular *P*, as a centre; therefore, a point *A* without a plane being given, to find the point *P* where the perpendicular *A* meets this plane, take three points *B*, *C*, *D*, equally distant from the point *A*, and find the centre of the circle which passes through these points, this centre will be the point *P* required.

331. *Scholium.* The angle *ABP* is called the *inclination of the oblique line* AB *to the plane* MN. It is manifest that this inclination is the same for all the oblique lines *AB*, *AC*, *AD*, &c., which depart equally from the perpendicular; for all the triangles *ABP*, *ACP*, *ADP*, &c., are equal.

THEOREM.

185. 332. *Let* AP (fig. 185) *be a perpendicular to the plane* MN, *and* BC *a line situated in this plane; if from the foot* P *of the perpendicular a line* PD *be drawn perpendicular to* BC, *and* AD *be joined,* AD *will be perpendicular to* BC.

Demonstration. Take $DB = DC$, and join *PB*, *PC*, *AB*, *AC*. Since $DB = DC$, the oblique line $PB = PC$; and, because $PB = PC$, the oblique lines *AB*, *AC*, considered with reference to the perpendicular *AP*, are equal (329); hence the line *AD* has two points *A* and *D* each equally distant from the extremities *B* and *C*; therefore *AD* is perpendicular to *BC* (55).

333. *Corollary.* It will be seen, at the same time, that *BC* is perpendicular to the plane *APD*, since *BC* is perpendicular at the same time to the two straight lines *AD* and *PD*.

334. *Scholium.* The two lines *AE*, *BC*, present an example of two lines which do not meet, because they are not situated in the same plane. The least distance of these lines is the straight line *PD*, which is at the same time perpendicular to the line *AP* and to the line *BC*. The distance *PD* is the shortest; because, if we join two other points, as *A* and *B*, we shall have $AB > AD$, $AD > PD$, and, for a still stronger reason, $AB > PD$.

The two lines *AE*, *CB*, although not situated in the same plane, are considered as making a right angle with each other, because *AE* and a line drawn through any point in it parallel to *BC*, would make a right angle with each other. In like manner, the line *AB* and the line *PD*, which represent two straight lines not situated in the same plane, are considered as making the same angle with each other, as is made by *AB* and a line parallel to *PD* drawn through some point in *AB*.

THEOREM.

335. *If the line* AP (fig. 186) *is perpendicular to the plane* MN, *every line* DE *parallel to* AP *will be perpendicular to the same plane.* Fig. 186.

Demonstration. Let there be a plane passing through the parallels *AP*, *DE*, intersecting the plane *MN* in *PD*; in the plane *MN* draw *BC* perpendicular to *PD*, and join *AD*.

According to the corollary of the preceding theorem *BC* is perpendicular to the plane *APDE*; consequently the angle *BDE* is a right angle; but the angle *EDP* is also a right angle, since *AP* is perpendicular to *PD*, and *DE* is parallel to *AP* (65); hence the line *DE* is perpendicular to each of the lines *DP*, *DB*; therefore it is perpendicular to the plane *MN* passing through them (325).

336. *Corollary* I. Conversely, if the straight lines *AP*, *DE*, are perpendicular to the same plane *MN*, they will be parallel; for, if they are not, through the point *D* draw a line parallel to *AP*; this parallel will be perpendicular to the plane *MN*, consequently there would be two perpendiculars to the same plane drawn through the same point, which is impossible (328).

337. *Corollary* II. Two lines *A* and *B*, parallel to a third *C*, are parallel to each other; for, let there be a plane perpendicular to the line *C*, the lines *A* and *B* parallel to this perpendicu-

lar will be perpendicular to the same plane; therefore, by the above corollary, they are parallel to each other.

It is supposed that the three lines are not in the same plane, without which the proposition would already be known (68).

THEOREM.

Fig. 187. 333. *If the straight line* AB (fig. 187) *is parallel to another straight line* CD, *drawn in the plane* MN, *it will be parallel to this plane.*

Demonstration. If the line *AB*, which is in the plane *ABCD*, should meet the plane *MN*, this can take place only in some point of the line *CD*, the common intersection of the two planes; now *AB* cannot meet *CD*, because it is parallel to it; consequently it cannot meet the plane *MN*; therefore it is parallel to this plane (314).

THEOREM.

Fig. 188. 339. *Two planes* MN, PQ (fig. 188), *perpendicular to the same straight line* AB, *are parallel to each other.*

Demonstration. If they can meet, let *O* be one of the common points of intersection, and join *OA*, *OB*; the line *AB*, perpendicular to the plane *MN*, is perpendicular to the straight line *OA* drawn through its foot in this plane; for the same reason, *AB* is perpendicular to *BO*; hence *OA*, *OB*, would be two perpendiculars let fall from the same point *O* upon the same straight line, which is impossible; consequently the planes *MN*, *PQ*, cannot meet; therefore they are parallel.

THEOREM.

Fig. 189. 340. *The intersections* EF, GH (fig. 189), *of two parallel planes* MN, PQ, *by a third plane* FG, *are parallel.*

Demonstration. If the lines *EF*, *GH*, situated in the same plane, are not parallel, being produced they will meet; consequently the planes *MN*, *PQ*, in which they are, would meet; therefore they would not be parallel.

THEOREM.

Fig. 188. 341. *The line* AB (fig. 188), *perpendicular to the plane* MN, *is perpendicular to the plane* PQ, *parallel to the plane* MN.

Demonstration. In the plane *PQ* draw at pleasure the line *BC*, and through *AB*, *BC*, suppose a plane *ABC* to pass intersecting the plane *MN* in *AD*, the intersection *AD* will be parallel to *BC* (340); but the line *AB*, perpendicular to the plane *MN*, is perpendicular to the straight line *AD*; consequently it will be perpendicular to its parallel *BC*; and, since the line *AB* is perpendicular to every line *BC* drawn through the foot of it in the plane *PQ*, it follows that it is perpendicular to the plane *PQ*.

THEOREM.

342. *The parallels* EG, FH (fig. 189), *comprehended between two parallel planes* MN, PQ, *are equal.* Fig. 189.

Demonstration. Through the parallels *EG*, *FH*, suppose a plane *EGHF* to pass meeting the parallel planes in *EF*, *GH*. The intersections *EF*, *GH*, are parallel (340) as well as *EG*, *FH*; consequently the figure *EGHF* is a parallelogram; therefore $EG = FH$.

343. *Corollary.* It follows from this, that *two parallel planes are throughout at the same distance from each other;* for, if *EG*, *FH*, are perpendicular to the two planes *MN*, *PQ*, they are parallel to each other (335); therefore they are equal.

THEOREM.

344. *If two angles* CAE, DBF (fig. 190) *not in the same plane, have their sides parallel and directed the same way, these angles will be equal, and their planes will be parallel.* Fig. 190.

Demonstration. Take $AC = BD$, $AE = BF$, and join *CE*, *DF*, *AB*, *CD*, *EF*. Since *AC* is equal and parallel to *BD*, the figure *ABDC*, is a parallelogram (87); therefore *CD* is equal and parallel to *AB*. For a similar reason, *EF* is equal and parallel to *AB*; consequently *CD* is also equal and parallel to *EF*; hence the figure *CEFD* is a parallelogram, and thus the side *CE* is equal and parallel to *DF*; the triangles then *CAE*, *DBF*, are equilateral with respect to each other; therefore the angle

$$CAE = DBF.$$

Again, the plane *ACE* is parallel to the plane *BDF*; for, let us suppose the plane parallel to *DBF*, drawn through the point *A*, to meet the lines *CD*, *EF*, in points different from *C* and *E*, for example, in *G* and *H*; then, according to article 342, the three lines *AB*, *GD*, *FH*, will be equal; but the three *AB*, *CD*, *EF*, are also equal; hence we should have $CD = GD$, and $FH = FE$, which is absurd; therefore the plane *ACE* is parallel to *BDF*.

345. *Corollary.* If two parallel planes *MN*, *PQ*, are met by two other planes *CABD*, *EABF*, the angles *CAE*, *DBF*, formed by the intersections in the parallel planes, are equal; for the intersection *AC* is parallel to *BD* (340), and *AE* to *BF*, therefore the angle $CAE = DBF$.

THEOREM.

Fig. 190. 346. *If three straight lines not in the same plane* AB, CD, EF (fig. 190), *are equal and parallel, the triangles* ACE, BDF, *formed by joining the extremities of these lines, on the one hand and on other, will be equal and their planes will be parallel.*

Demonstration. Since *AB* is equal and parallel to *CD*, the figure *ABDC* is a parallelogram; consequently the side *AC* is equal and parallel to *BD*. For a similar reason the sides *AE*, *BF*, are equal and parallel, as also *CE*, *DF*; hence the two triangles *ACE*, *BDF*, are equal; it may be shown moreover, as in the preceding proposition, that their planes are parallel.

THEOREM.

347. *Two straight lines comprehended between three parallel planes are divided into parts that are proportional to each other.*

Fig. 191. *Demonstration.* Let us suppose that the line *AB* (*fig.* 191) meets the parallel planes *MN*, *PQ*, *RS*, in *A*, *E*, *B*, and that the line *CD* meets the same planes in *C*, *F*, *D*, we shall have

$$AE : EB :: CF : FD.$$

Draw *AD* meeting the plane *PQ* in *G*, and join *AC*, *EG*, *GF*, *BD*; the intersections *EG*, *BD*, of the parallel planes *PQ*, *RS*, by the plane *ABD*, are parallel (340); hence, $AE : EB :: AG : GD$; and, because the intersections *AC*, *GF*, are parallel,

$$AG : GD :: CF : FD;$$

therefore, on account of the common ratio, $AG : GD$, we have

$$AE : EB :: CF : FD.$$

THEOREM.

Fig. 192. 348. *Let* ABCD (fig. 192) *be any quadrilateral either in the same plane or not; if the opposite sides are cut proportionally by* *two straight lines* EF, GH, *so that* $AE : EB :: DF : FC$, and $BG : GC :: AH : HD$, *the straight lines* EF, GH, *will cut each other in a point* M, *in such a manner that* $HM : MG :: AE : EB$, $EM : MF :: AH : HD$.

Demonstration. Let there be any plane $AbHcD$ passing through AD which does not pass through GH; through the points E, B, C, F, draw Ee, Bb, Cc, Ff, parallel to GH meeting this plane in e, b, c, f. On account of the parallels Bb, GH, Cc,

$$bH : Hc :: BG : GC :: AH : HD \quad (196);$$

consequently the triangles AHb, DHc, are similar (208). Also

$$Ae : eb :: AE : EB$$

and $$Df : fc :: DF : FC,$$

hence $$Ae : eb :: Df : fc,$$

or, by *composition* $$Ae : Df :: Ab : Dc;$$

but, on account of the similar triangles AHb, DHc,

$$Ab : Dc :: AH : HD,$$

consequently $$Ae : Df :: AH : HD.$$

Besides, the triangles AHb, DHc, being similar, the angle $HAe = HDf$; hence the triangles AHe, DHf, are similar (208), and consequently the angle $AHe = DHf$. It follows then, in the first place, that eHf is a straight line, and that thus the three parallels Ee, GH, Ff, are situated in the same plane which contains the two straight lines EF, GH; therefore *these must cut each other in a point* M. Moreover, on account of the parallels Ee, MH, Ff, $EM : MF :: eH : Hf :: AH : HD$.

By a similar construction, referred to the side AB, it may be demonstrated that $HM : MG :: AE : EB$.

THEOREM.

349. *The angle comprehended between two planes* MAN, MAP, *may be measured, conformably to the definition, by the angle* PAN (fig. 193) *made by the two lines* AN, AP, *drawn one in one of these planes and the other in the other perpendicularly to the common intersection* AM.

Fig. 193.

Demonstration. In order to show the legitimacy of this measure it is necessary to prove, 1. that it is constant, or in other words, that it is the same to whatever point of the common intersection the two perpendiculars are drawn.

If we take another point M, and draw MC in the plane MN, and MB in the plane MP, perpendicular to the common intersection *AM*; since *MB* and *AP* are perpendicular to the same line *AM*, they are parallel to each other. For the same reason *MC* is parallel to *AN*; consequently the angle *BMC* = *PAN* (344); therefore, whether the perpendiculars be drawn to the point *M* or to the point *A*, the angle is always the same.

2. It is necessary to show that, if the angle of the two planes increases or diminishes, the angle *PAN* increases and diminishes in the same ratio.

In the plane *PAN* describe, from the centre *A* and with any radius, the arc *NDP*, and from the centre *M* and with the same radius, the arc *CEB*; draw *AD* to any point *D* in the arc *NP*; the two planes *PAN*, *BMC*, being perpendicular to the same straight line *MA* are parallel to each other (339); consequently the intersections *AD*, *ME*, of the two planes by the third *AMD*, are parallel; therefore the angle *BME* is equal to *PAD* (344).

Calling, for the present, the angle formed by the two planes *MP*, *MN*, a *wedge*, if the angle *DAP* were equal to *DAN*, it is evident that the wedge *DAMP* would be equal to the wedge *DAMN*; for the base *PAD* might be applied exactly to its equal *DAN*, and the altitude *AM* would be the same for both; therefore the two wedges would coincide with each other. It is manifest, likewise, if the angle *DAP* were contained a certain number of times without a fraction in the angle *PAN*, the wedge *DAMP* would be contained as many times in the wedge *PAMN*. Moreover, from a ratio in an entire number to any ratio whatever the conclusion is legitimate, and has been demonstrated in a case altogether similar (122); consequently, whatever be the ratio of the angle *DAP* to the angle *PAN*, the wedge *DAMP* will have the same ratio to the wedge *PAMN*; therefore the angle *NAP* may be taken for the measure of the wedge *PAMN*, or of the angle made by the two planes *MAP*, *MAN*.

350. *Scholium.* It is with angles formed by two planes, as it is with angles formed by two straight lines. Thus, when two planes intersect each other, the angles opposite to each other at the vertex are equal, and the adjacent angles are together equal to two right angles; therefore, when one plane is perpendicular to another, the latter is perpendicular to the former. Also, when two parallel planes are intersected by a third plane, the same properties exist with respect to the angles thus formed, as take place, when two parallel lines are met by a third line (64).

THEOREM.

351. *The line* AP (fig. 194) *being perpendicular to the plane* MN, *any plane* APB, *passing through* AP, *will be perpendicular to the plane* MN. Fig. 194.

Demonstration. Let *BC* be the intersection of the planes *AB*, *MN*; if, in the plane *MN*, the line *DE* be drawn perpendicular to *BP*, the line *AP*, being perpendicular to the plane *MN*, will be perpendicular to each of the two straight lines *BC*, *DE*. But the angle *APD* formed by the two perpendiculars *PA*, *PD*, at the common intersection *BP*, measures the angle of the two planes *AB*, *MN*; therefore, since this angle is a right angle, the two planes are perpendicular to each other (317).

352. *Scholium.* When three straight lines, as *AP*, *BP*, *DP*, are perpendicular to each other, each of these lines is perpendicular to the plane of the two others, and the three planes are perpendicular to each other.

THEOREM.

353. *If the plane* AB (fig. 194) *is perpendicular to the plane* MN, *and in the plane* AB *the line* AP *be drawn perpendicular to the common intersection* PB, *the line* AP *will be perpendicular to the plane* MN. Fig. 194.

Demonstration. If, in the plane *MN*, the line *PD* be drawn perpendicular to *PB*, the angle *APD* will be a right angle, since the planes are perpendicular to each other; consequently, the line *AP* is perpendicular to the two straight lines *PB*, *PD*; therefore it is perpendicular to the plane *MN*.

354. *Corollary.* If the plane *AB* is perpendicular to the plane *MN*, and if through a point *P* of the common intersection a perpendicular to the plane *MN* be drawn, this perpendicular will be in the plane *AB*; for, if it is not, there may be drawn, in the plane *AB*, a line *AP* perpendicular to the common intersection *BP*, which would be at the same time perpendicular to the plane *MN*; therefore there would be two perpendiculars to the plane *MN* at the same point *P*, which is impossible (325).

THEOREM.

355. *If two planes* AB, AD (fig. 194), *are perpendicular to a third* MN, *their common intersection* AP *will be perpendicular to this third plane.* Fig. 194.

Demonstration. If through the point P a perpendicular to the plane MN be drawn, this perpendicular must be at the same time in the plane AB and in the plane AD (354); therefore it is their common intersection AP.

THEOREM.

356. *If a solid angle is formed by three plane angles, the sum of either two of these angles will be greater than the third.*

Fig. 195. *Demonstration.* We need consider only the case in which the plane angle to be compared with the two others is greater than either of them. Let there be then the solid angle S (*fig.* 195) formed by the three plane angles ASB, ASC, BSC, and let us suppose that the angle ASB is the greatest of the three; we say that $ASB < ASC + BSC$.

In the plane ASB make the angle $BSD = BSC$, draw at pleasure the straight line ADB; and, having taken $SC = SD$, join AC, BC.

The two sides BS, SD, are equal to the two BS, SC, and the angle $BSD = BSC$; hence the two triangles BSD, BSC, are equal; consequently $BD = BC$. But $AB < AC + BC$; if we take from the one BD, and from the other its equal BC, there will remain $AD < AC$. The two sides AS, SD, are equal to the two AS, SC, and the third AD is less than the third AC; therefore the angle $ASD < ASC$ (42). Adding $BSD = BSC$, we shall have

$$ASD + BSD \text{ or } ASB < ASC + BSC.$$

THEOREM.

357. *The sum of the plane angles which form a solid angle is always less than four right angles.*

Fig. 196. *Demonstration.* Suppose the solid angle S (*fig.* 196) to be cut by a plane $ABCDE$; from a point O taken in this plane draw to the several angles the lines OA, OB, OC, OD, OE.

The sum of the angles of the triangles ASB, BSC, &c., formed about the vertex S, is equal to the sum of the angles of an equal number of triangles AOB, BOC, &c., formed about the vertex O. But, at the point B, the angles ABO, OBC, taken together, make the angle ABC less than the sum of the angles ABS, SBC (356); likewise, at the point C, $BCO + OCD < BCS + SCD$, and so on through all the angles of the polygon $ABCDE$. It follows then, that of the triangles whose vertex is in O the sum of the angles at the bases is less than the sum of the angles at the bases of the triangles whose vertex is in S. Hence, the sum of the angles about the point O is greater than the sum of the angles about the point S. But the sum of the angles about the point O is equal to four right angles (34); therefore the sum of the plane angles, which form a solid angle S, is less than four right angles.

358. *Scholium.* It is supposed, in this demonstration, that the solid angle is convex, or that the plane of neither of the faces would, by being produced, cut the solid angle; if it were otherwise, the sum of the plane angles would no longer be limited, and might be of any magnitude whatever.

THEOREM.

359. *If two solid angles are respectively contained by three plane angles which are equal, each to each, the planes of any two of these angles in the one will have the same inclination to each other as the planes of the homologous angles in the other.*

Demonstration. Let the angle $ASC = DTF$ (*fig.* 197), the angle $ASB = DTE$, and the angle $BSC = ETF$; we say that the two planes ASC, ASB, will have, with respect to each other, an inclination equal to that of the planes DTF, DTE. Fig. 197.

Take SB of any magnitude, and draw BO perpendicular to the plane ASC; from the point O, where this perpendicular meets the plane, draw OA, OC, perpendicular respectively to SA, SC; join AB, BC. Take also $TE = SB$; and draw EP perpendicular to the plane DTF; from the point P draw PD, PF, perpendicular respectively to TD, TF; and join ED, EF.

The triangle SAB is right-angled at A, and the triangle TDE at D (332); and, since the angle $ASB = DTE$, we have also $SBA = TED$. Moreover, $SB = TE$; therefore the triangle $SAB = TDE$; consequently $SA = TD$, and $AB = DE$. It may be shown, in a similar manner, that $SC = TF$, and $BC = EF$. This being supposed, the quadrilateral $SAOC$ is equal to the quadrilateral $TDPF$; for, if we apply the angle ASC to its equal DTF, because $SA = TD$, and $SC = TF$, the point A will fall upon D, and the point C upon F. At the same time AO, perpendicular to SA, will fall upon DP, perpendicular to TD; and, in like manner, OC upon PF; therefore the point O will fall upon the point P, and we shall have $AO = DP$. But the triangles AOB, DPE, are right-angled at O, and P, the hypothenuse $AB = DE$

and the side $AO = DP$; consequently the triangles are equal (56); hence $OAB = PDE$. But the angle OAB is the inclination of the two planes ASB, ASC; and the angle PDE is the inclination of the two planes DTE, DTF; therefore these two inclinations are equal to each other.

It should be observed, however, that the angle A of the right-angled triangle OAB is not properly the inclination of the two planes ASB, ASC, except when the perpendicular BO falls, with respect to SA, on the same side as SC; if it should fall on the other side, the angle of the two planes would be obtuse, and, added to the angle A of the triangle OAB, it would make two right angles. But, in the same case, the angle of the two planes TDE, TDF, would be likewise obtuse, and, added to the angle D of the triangle DPE, it would make two right angles; therefore, as the angle A would be always equal to D, we infer, in like manner, that the inclination of the two planes ASB, ASC, is equal to that of the two planes TDE, TDF.

360. *Scholium.* If two solid angles are respectively contained by three plane angles which are equal, each to each, and if, at the same time, the angles of the one are *disposed in the same manner* as the homologous angles of the other, these solid angles will be equal, and, being applied the one to the other, will coincide. Indeed, we have already seen that the quadrilateral $SAOC$ may be placed upon its equal $TDPF$; thus, by placing SA upon TD, SC would fall upon TF, and the point O upon the point P. But, on account of the equality of the triangles AOB, DPE, the line OB perpendicular to the plane ASC is equal to PE perpendicular to the plane TDF; moreover the perpendiculars are directed the same way; therefore the point B will fall upon the point E, the line SB upon TE, and the two solid angles will coincide entirely with each other.

This coincidence, however, does not take place except by supposing that the plane angles *are disposed in the same manner* in each of the two solid angles; for if the plane angles were disposed *in a contrary order* in the one from what they are in the other; or, which comes to the same thing, if the perpendiculars OB, PE, instead of being directed the same way with respect to the planes ASC, DTF, were directed contrary ways, it would be impossible to make the solid angles coincide with each other. Still it would not be the less true, that, agreeably to the theorem, the planes of the homologous angles would be equally inclined to each other; so that the two solid angles would be equal in all their constituent parts, without the property however of coinciding, when applied the one to the other. This kind of equality, which is not absolute, or does not admit of superposition, deserves to be distinguished by a particular denomination; we shall call it *equality by symmetry*.

Thus the two solid angles under consideration; which are respectively contained by three plane angles equal, each to each, but disposed in a contrary order in the one from what they are in the other, we shall call *angles equal by symmetry*, or simply *symmetrical angles*.

The same remark is applicable to solid angles contained by more than three plane angles; thus a solid angle contained by the plane angles A, B, C, D, E, and another solid angle contained by the same angles in the inverse order A, E, D, C, B, may be such that the planes of the homologous angles shall be equally inclined to each other. These two solid angles, which would be equal without admitting of superposition, we shall call *solid angles equal by symmetry*, or *symmetrical solid angles*.

There is not properly an equality by symmetry among plane figures; all those to which we might give this name, have the property of absolute equality, or equality by superposition. The reason is, that a plane figure may be reversed, and the upper side be taken for the under. It is not so with respect to solids, in which the third dimension may be taken in two different ways.

PROBLEM.

361. *Three plane angles forming a solid angle being given, to find, by a plane construction, the angle which two of these planes make with each other.*

Solution. Let S (*fig.* 198) be the proposed solid angle in which the three plane angles ASB, ASC, BSC, are known; the angle made by two of these planes with each other, ASB, ASC, for example, is required. Fig. 198.

The same construction being supposed as in the preceding theorem, the angle OAB would be the angle sought. It is proposed to find the same angle by a plane construction, or by lines traced upon a plane.

In order to this, make upon a plane the angles *B'SA*, *ASC*, *B''SC*, equal to the angles *BSA*, *ASC*, *BSC*, in the solid figure; take *B'S*, *B''S*, each equal to *BS* in the solid figure; from the points *B'* and *B''* let fall *B'A* and *B''C* perpendicularly upon *SA* and *SC*, which will meet in a point *O*. From the point *A*, as a centre, and with the radius *AB* describe the semicircumference *B' b E*; at the point *O* erect upon *B'E* the perpendicular *O b* meeting the circumference in *b*; join *A b*, and the angle *EA b* will be the inclination sought of the two planes *ASC*, *ASB*, in the solid angle.

We have only to show that the triangle *AOb* of the plane figure is equal to the triangle *AOB* of the solid figure. Now the two triangles *B'SA*, *BSA*, are right-angled at *A*, and the angles at *S* are equal, consequently the angles at *B* and *B'* are also equal. But the hypothenuse *SB'* is equal to the hypothenuse *SB*; therefore the triangles are equal; hence *SA* in the plane figure is equal to *SA* in the solid figure, also *AB'* or its equal *A b* in the plane figure, is equal to *AB* in the solid figure. It may be shown, in the same manner, that *SC* in one figure is equal to *SC* in the other; whence it follows that the quadrilateral *SAOC* in one figure is equal to *SAOC* in the other, and that thus *AO* in the plane figure is equal to *AO* in the solid figure; consequently the right-angled triangles *AO b*, *AOB*, have their hypothenuses equal and one side of the one equal to one side of the other; they are therefore equal, and the angle *EA b*, found by the plane construction, is equal to the inclination of the planes *SAB*, *SAC*, in the solid angle.

When the point *O* falls between *A* and *B'* in the plane figure, the angle *EA b* becomes obtuse, and always measures the true inclination of the planes. It is on this account that we have designated the required inclination by *EA b*, and not by *OA b*, in order that the same solution may be adapted to every case without exception.

362. *Scholium.* It may be asked, if, any three plane angles, taken at pleasure, can be made to form a solid angle.

In the first place, it is necessary that the sum of the three given angles should be less than four right angles, otherwise the solid angle could not be formed (357); it is necessary, moreover, that, after having taken two of the angles at pleasure *B'SA*, *ASC*, the third *CSB''* should be such that the perpendicular

$B''C$ to the side SC shall meet the diameter $B'E$ between its extremities B' and E. Thus the limits of the magnitude of the angle CSB'' are such as require the perpendicular $B''C$ to terminate at the points B' and E. From these points let fall upon CS the perpendiculars $B'I$, EK, meeting in I and K the circumference described upon the radius SB'', and the limits of the angle CSB'' will be CSI and CSK.

But, in the isosceles triangle $B'SI$, the line CS produced being perpendicular to the base $B'I$,

$$\text{the angle } CSI = CSB' = ASC + ASB'.$$

And, in the isosceles triangle ESK the line SC being perpendicular to EK, the angle $CSK = CSE$. Moreover, on account of the equal triangles ASE, ASB', the angle $ASE = ASB'$; therefore CSE or $CSK = ASC - ASB'$.

Hence we infer that the problem will be possible, while the third angle is less than the sum of the two others ASC, ASB', and greater than their difference, a condition which accords with the theorem art. 356; for, by this theorem, we must have $CSB'' < ASC + ASB'$, also $ASC < CSB'' + ASB'$, or

$$CSB'' > ASC - ASB'.$$

PROBLEM.

363. *Two of the three plane angles, which form a solid angle, being given together with the angle which their planes make with each other, to find the third plane angle.*

Solution. Let ASC, ASB' (*fig.* 198), be the two given plane angles, and let us suppose, for the present, that CSB'' is the third angle sought; then, by constructing the figure as in the preceding problem, the angle contained by the planes of the two first would be EAb. Now, as we determine the angle EAb by means of CSB'', the two others being given; so we can determine CSB'' by means of EAb, and thus solve the proposed problem. Fig. 198.

Having taken SB' of any magnitude at pleasure, let fall upon SA the indefinite perpendicular $B'E$, make the angle EAb equal to the angle of the two given planes; from the point b, where the side Ab meets the circumference described with the centre A and the radius AB', let fall upon AE the perpendicular bO, and from the point O let fall upon SC the indefinite perpendicular OCB'', which terminate in B'' making $SB'' = SB'$; the angle CSB'' will be the third plane required.

For, if a solid angle be formed of the three planes $B'SA$, ASC, CSB'', the inclination of the planes containing the given angles ASB', ASC, will be equal to the given angle $EA\,b$.

364. *Scholium.* If a solid angle is *quadruple*, or formed by four plane angles ASB, BSC, CSD, DSA (*fig.* 199), we cannot, by knowing these angles, determine the mutual inclination of their planes; for with the same plane angles any number of solid angles may be formed. But, if a condition be added, if, for example, the inclination of the two planes ASB, BSC, be given, then the solid angle is entirely determinate, and the inclination of any two of the planes may be found. Suppose a *triple* solid angle formed by the plane angles ASB, BSC, ASC; the two first angles are given as well as the inclination of their planes; we can then, by the problem just resolved, determine the third angle ASC. Afterward, if we consider the triple solid angle formed by the plane angles ASC, ASD, DSC, these three angles are known; thus the solid angle is entirely determinate. But the quadruple solid angle is formed by the union of the two triple solid angles of which we have been speaking; therefore, since these partial angles are known and determinate, the whole angle will be known and determinate. Fig. 199.

The angle of the two planes ASD, DSC, may be found immediately by means of the second partial solid angle. As to the angle of the two planes BSC, CSD, it is necessary in one of the partial solid angles to find the angle comprehended between the two planes ASC, DSC, and in the other the angle comprehended between the two planes ASC, BSC; the sum of these angles will be the angle comprehended between the two planes BSC, DSC.

It will be found, in the same manner, that, in order to determine a quintuple solid angle, it is necessary to know, beside the five plane angles which compose it, two of the mutual inclinations of their planes; in a sextuple solid angle it is necessary to know three of these inclinations, and so on.

SECTION SECOND.

Of Polyedrons.

DEFINITIONS.

365. EVERY solid terminated by planes or plane faces is called a *solid polyedron*, or simply a *polyedron*. These planes are themselves necessarily terminated by straight lines.

A solid of four faces is called a *tetraedron*, one of six a *hexaedron*, one of eight an *octaedron*, one of twelve a *dodecaedron*, one of twenty an *icosaedron*, &c.

The tetraedron is the most simple of polyedrons; for it requires at least three planes to form a solid angle, and these three planes would leave an opening, to close which a fourth plane is necessary.

366. The common intersection of two adjacent faces of a polyedron is called a *side* or *edge* of the polyedron.

367. A *regular polyedron* is one, all whose faces are equal regular polygons, and all whose solid angles are equal to each other. There are five polyedrons of this kind.

368. A *prism* is a solid comprehended under several parallelograms terminated by two equal and parallel polygons.

To construct this solid let *ABCDE* (*fig.* 200) be any polygon, if, in a plane parallel to *ABC* we draw the lines *FG*, *GH*, *HI*, &c.. equal and parallel to the sides *AB*, *BC*, *CD*, &c., we shall form the polygon *FGHIK* equal to *ABCDE*; if now we connect the vertices of the homologous angles by the straight lines *AF*, *BG*, *CH*, &c., the faces *ABGF*, *BCHG*, &c., will be parallelograms, and the solid thus formed *ABCDEFGHIK* will be a prism. Fig. 200.

369. The equal and parallel polygons *ABCDE*, *FGHIK*, are called the *bases of the prism*. The other planes taken together, constitute the *lateral* or *convex surface of the prism*. The equal straight lines *AF*, *BG*, *CH*, &c., are called the *sides* of the prism.

370. The *altitude of a prism* is the distance between its bases, or the perpendicular let fall from a point in the superior base upon the plane of the inferior.

371. A *right prism* is one whose sides *AF*, *BG*, &c., are perpendicular to the planes of the bases; in this case, each of the

sides is equal to the altitude of the prism. In every other case the prism is *oblique*, and the altitude is less than the side.

372. A *prism* is *triangular*, *quadrangular*, *pentagonal*, *hexagonal*, &c., according as the base is a triangle, a quadrilateral, a pentagon, a hexagon, &c.

Fig. 206. 373. A prism whose base is a parallelogram (*fig.* 206), has all its faces parallelograms, and is called a *parallelopiped.*

A *parallelopiped* is *rectangular*, when all its faces are rectangles.

374. Among rectangular parallelopipeds is distinguished the *cube* or regular hexaedron comprehended under six equal squares.

375. A *pyramid* is a solid formed by several triangular planes proceeding from the same point and terminating in the sides of a polygon *ABCDE* (*fig.* 196). Fig. 196.

The polygon *ABCDE* is called the base of the pyramid, the point *S* its vertex, and the triangles *ASB*, *BSC*, &c., compose the *lateral* or *convex surface* of the pyramid.

376. The *altitude* of a pyramid is the perpendicular let fall from the vertex upon the plane of the base, produced if necessary.

377. A pyramid is *triangular*, *quadrangular*, &c., according as the base is a triangle, a quadrilateral, &c.

378. A pyramid is *regular*, when the base is a regular polygon, and the perpendicular, let fall from the vertex to the plane of the base, passes through the centre of this base. This line is called the *axis* of the pyramid.

379. The *diagonal* of a polyedron is a straight line which joins the vertices of two solid angles not adjacent.

380. I shall call *symmetrical polyedrons* two polyedrons which, having a common base, are similarly constructed, the one above the plane of this base and other below it, with this condition, that the vertices of the homologous solid angles be situated at equal distances from the plane of the base, in the same straight line perpendicular to this plane.

Fig. 202. If the straight line *ST* (*fig.* 202), for example, is perpendicular to the plane *ABC*, and is bisected at the point *O*, where it meets this plane, the two pyramids *SABC*, *TABC*, which have the common base *ABC*, are two symmetrical polyedrons.

381. Two *triangular pyramids* are *similar* when they have two faces similar, each to each, similarly placed, and equally inclined to each other.

Thus, if we suppose the angle $ABC = DEF$, $BAC = EDF$, $ABS = DET$, $BAS = EDT$, (*fig.* 203), if also the inclination of the planes ABS, ABC, is equal to that of their homologous planes DTE, DEF, the pyramids $SABC$, $TDEF$, are similar. Fig. 203.

382. Having formed a triangle with the vertices of three angles, taken in the same face or base of a polyedron, we can imagine the vertices of the different solid angles of the polyedrons, situated out of the plane of this base, to be the vertices of as many triangular pyramids, which have for their common base the above triangle; and these several pyramids will determine the positions of the several solid angles of the polyedron with respect to the base. This being supposed;

Two *polyedrons* are *similar*, when, the bases being similar, the vertices of the homologous solid angles are determined by triangular pyramids similar each to each.

383. I shall call *vertices* of a *polyedron* the points situated at the vertices of the different solid angles.

N. B. We shall consider only those polyedrons, which have saliant angles, or *convex* polyedrons. We thus denominate those, the surface of which cannot be met by a straight line in more than two points. In polyedrons of this description the plane of neither of the faces can, by being produced, cut the solid; it is impossible then, that the polyedron should be in part above the plane of one of the faces and in part below it; it is wholly on one side of this plane.

THEOREM.

384. *Two polyedrons cannot have the same vertices, the number also being the same, without coinciding the one with the other.*

Demonstration. Let us suppose one of the polyedrons already constructed, if we would construct another having the same vertices, the number also being the same, it is necessary that the planes of this last should not all pass through the same points as in the first; if they did, they would not differ the one from the other; but then it is evident that any new planes would cut the first polyedron; there would then be vertices above these planes and vertices below them, which does not consist with a convex polyedron; therefore, if two polyedrons have the same vertices, the number also being the same, they must necessarily coincide the one with the other.

Fig 204. 385. *Scholium.* The points A, B, C, K, &c. (*fig.* 204), being given in position to be used as the vertices of a polyedron, it is easy to describe the polyedron.

Take, in the first place, three neighbouring points D, E, H, such that the plane DEH shall pass, if there is occasion for it, through other points K, C, but leaving all the rest on the same side, all above the plane, or all below it; the plane DEH or $DEHKC$, thus determined, will be a face of the solid. Through one of the sides EH of this face, suppose a plane to pass, and to turn upon this line until it meets a new vertex F, or several at the same time F, I; we shall thus have a second face FEH or $FEHI$. Proceed in this manner, by making planes to pass through the sides of the faces, until the solid is terminated in all directions; this solid will be the polyedron required, for there are not two which can have the same vertices.

THEOREM.

386. *In two symmetrical polyedrons the homologous faces are equal, each to each, and the inclination of two adjacent faces in one of the solids is equal to the inclination of the homologous faces in the other.*

Fig. 205. *Demonstration.* Let $ABCDE$ (*fig.* 205) be the common base of the two polyedrons, M and N the vertices of any two solid angles of one of the polyedrons, M' and N' the homologous vertices of the other polyedron; the straight lines MM', NN', must be perpendicular to the plane ABC, and be bisected at the points m and n (380), where they meet this plane. This being supposed, we say that the distance MN is equal to $M'N'$.

For, if the trapezoid $mM'N'n$ be made to revolve about mn, until its plane comes into the position of the plane $mMNn$, on account of the right angles at m and n, the side mM' will fall upon its equal mM, and nN' upon nN; therefore the two trapezoids will coincide, and we shall have $MN = M'N'$.

Let P be a third vertex in the superior polyedron, and P' the homologous vertex in the other, we shall have, in like manner, $MP = M'P'$, and $NP = N'P'$; consequently *the triangle* MNP, *formed by joining any three vertices of the superior polyedron is equal to the triangle* M'N'P', *formed by joining the homologous vertices of the other polyedron.*

If, among these triangles we consider only those which are formed at the surface of the polyedrons, we can conclude already that the surfaces of the two polyedrons are composed of the same number of triangles equal, each to each.

We say now that, if some of these triangles are in the same plane upon one surface and form the same polygonal face, the homologous triangles will be in the same plane upon the other surface and will form an equal polygonal face.

Let MPN, NPQ, be two adjacent triangles, which we suppose in the same plane, and let $M'P'N'$, $N'P'Q'$, be the homologous triangles. We have the angle $MNP = M'N'P'$, the angle $PNQ = P'N'Q'$; and, if we were to join MQ and $M'Q'$, the triangle MNQ would be equal to $M'N'Q'$, thus we should have the angle $MNQ = M'N'Q'$. But, since $MPNQ$ is one plane, we have the angle $MNQ = MNP + PNQ$; we have also

$$M'N'Q' = M'N'P' + P'N'Q'.$$

Now, if the three planes $M'N'P'$, $P'N'Q'$, $M'N'Q'$, are not confounded in one, they will form a solid angle, and we shall have the angle $M'N'Q' < M'N'P' + P'N'Q'$ (356); therefore, as this condition does not exist, the two triangles $M'N'P'$, $P'N'Q'$, are in the same plane.

We hence infer that each face, whether triangular, or polygonal, in one polyedron, corresponds to an equal face in the other, and that thus the two polyedrons are comprehended under the same number of planes equal, each to each.

It remains to show that the inclination of any two adjacent faces in one of the polyedrons is equal to the inclination of the two homologous faces in the other.

Let MPN, NPQ, be two triangles formed upon the common edge NP in the planes of two adjacent faces; let $M'P'N'$, $N'P'Q'$, be the homologous triangles. We can conceive at N a solid angle formed by the three plane angles MNQ, MNP, PNQ, and at N' a solid angle formed by the three $M'N'Q'$, $M'N'P'$, $P'N'Q'$. Now it has already been proved that these plane angles are equal, each to each; consequently the inclination of the two planes MNP, PNQ, is equal to that of their homologous planes $M'N'P'$, $P'N'Q'$ (359).

Therefore in symmetrical polyedrons the faces are equal, each to each, and the planes of any two adjacent faces of one of the solids have the same inclination to each other as the planes of the two homologous faces of the other solid.

387. *Scholium.* It may be remarked that *the solid angles of the one polyedron are symmetrical with the solid angles of the other;* for, if the solid angle N is formed by the planes MNP, PNQ, QNR, &c., its homologous angle N' is formed by the planes $M'N'P'$, $P'N'Q'$, $Q'N'R'$, &c. These last seem to be disposed in the same order as the others; but, as one of the solid angles is inverted with respect to the other, it follows that the actual disposition of the planes, which form the solid angle N' is the reverse of that which exists with respect to the homologous angle N. Moreover the inclinations of the successive planes in the one are equal respectively to those in the other; therefore these solid angles are symmetrical with respect to each other. See art. 360.

It will be perceived, from what has been said, that *any polyedron whatever can have only one polyedron symmetrical with it.* For, if there were constructed, upon another base, a new polyedron symmetrical with the given polyedron, the solid angles of this last would always be symmetrical with the angles of the given polyedron; consequently they would be equal to those of the symmetrical polyedron constructed upon the first base. Moreover, the homologous faces would always be equal; whence these two symmetrical polyedrons, constructed upon the one base and upon the other, would have their faces equal and their solid angles equal; therefore they would coincide by superposition, and would make one and the same polyedron.

THEOREM.

388. *Two prisms are equal, when three planes containing a solid angle of the one are equal to three planes containing a solid angle of the other, each to each, and are similarly placed.*

Fig. 200. *Demonstration.* Let the base $ABCDE$ (*fig.* 200), be equal to the base $abcde$, the parallelogram $ABGF$ equal to the parallelogram $abgf$, and the parallelogram $BCHG$ equal to the parallelogram $bchg$; we say that the prism $ABCI$ will be equal to the prism $abci$.

For, let the base $ABCDE$ be placed upon the base $abcde$, the two bases will coincide. But the three plane angles, which form the solid angle B, are equal to the three plane angles, which from the solid angle b, each to each, namely $ABC = abc$, $ABG = abg$, *GBC* = *g b c*; also these angles are similarly placed; therefore the solid angles *B* and *b* are equal (360), and consequently the side *BG* will fall upon its equal *b g*. We see also that, on account of the equal parallelograms *ABGF*, *a b g f*, the side *GF* will fall upon its equal *g f*, and likewise *GH* upon *g h*; therefore the superior base *FGHIK* will coincide entirely with its equal *f g h i k*, and the two solids will form one and the same solid, since they have the same vertices (384).

389. *Corollary. Two right prisms, which have equal bases and equal altitudes, are equal.* For, since the side *AB* = *a b*, and the altitude *BG* = *b g*, the rectangle *ABGF* = *a b g f*; the same may be proved with respect to the rectangles *BGHC*, *b g h c*; thus the three planes, which form the solid angle *B*, are equal to the three which form the solid angle *b*, therefore the two prisms are equal.

THEOREM.

390. *In every parallelopiped the opposite planes are equal and parallel.*

Demonstration. According to the definition of this solid, the bases *ABCD*, *EFGH* (*fig.* 206), are equal parallelograms, and their sides are parallel (373). It remains then to demonstrate that the same is true with respect to two opposite lateral faces, as *AEHD*, *BFGC*. Now *AD* is equal and parallel to *BC*, since the figure *ABCD* is a parallelogram; for a similar reason *AE* is equal and parallel to *BF*; consequently the angle *DAE* is equal to the angle *CBF* (344), and the plane *DAE* parallel to *CBF*; therefore also the parallelogram *DAEH* is equal to the parallelogram *CBFG*. In like manner it may be demonstrated that the opposite parallelograms *ABFE*, *DCGH*, are equal and parallel. Fig. 206.

391. *Corollary.* Since a parallelopiped is a solid comprehended under six planes of which the opposite ones are equal and parallel, it follows that either of the faces and its opposite may be taken for the bases of the parallelopiped.

392. *Scholium.* There being given three straight lines *AB*, *AE*, *AD*, passing through the same point *A*, and making given angles with each other; upon these three straight lines, a parallelopiped may be constructed; in order to this, a plane is to be made to pass through the extremity of each straight line parallel

to the plane of the two others; namely, through the point *B* a plane parallel to *DAE*, through the point *D* a plane parallel to *BAE*, and through the point *E* a plane parallel to *BAD*. The mutual meeting of these planes will form the parallelopiped required.

THEOREM.

393. *In every parallelopiped the opposite solid angles are symmetrical, and the diagonals drawn through the vertices of these angles bisect each other.*

Fig. 206. *Demonstration.* Let us compare, for example, the solid angle *A* (*fig.* 206) with the solid angle *G*; the angle *EAB*, equal to *EFB*, is also equal to *HGC*, the angle $DAE = DHE = CGF$, and the angle $DAB = DCB = HGF$; consequently the three plane angles which form the solid angle *A*, are equal to the three, which form the solid angle *G*, each to each; besides, it is evident that their disposition in the one is different from that in the other; therefore the two solid angles *A* and *G* are symmetrical (359).

Again, let us suppose the two diagonals *EC*, *AG*, to be drawn each through opposite vertices; since *AE* is equal and parallel to *CG*, the figure *AEGC* is a parallelogram; consequently the diagonals *EC*, *AG*, bisect each other. It may be demonstrated, in the same manner, that the diagonal *EC* and another *DF* also bisect each other; therefore the four diagonals bisect each other in a point which may be regarded as the centre of the parallelopiped.

THEOREM.

Fig. 207. 394. *The plane* BDHF (fig. 207), *which passes through two opposite parallel edges* BF, DH, *of a parallelopiped* AG *divides it into two triangular prisms* ABD-HEF, GHF-BCD, *symmetrical with each other.*

Demonstration. In the first place the solids are prisms; for the triangles *ABD*, *EFH*, having the sides of the one equal and parallel to those of the other, are equal; and, at the same time, the lateral faces *ABFE*, *ADHE*, *BDHF*, are parallelograms; therefore the solid *ABD-HEF* is a prism. The same may be proved with respect to the solid *GHF-BCD*. We say now that these two prisms are symmetrical with each other.

Upon the base *ABD* make the prism *ABD-E′F′H′* symmetrical with the prism *ABD-EFH.* According to what has been demonstrated (386), the plane *ABF′E′* is equal to *ABFE*, and the plane *ADH′E′* is equal to *ADHE*; but, if we compare the prism *GHF-BCD* with the prism *ABD-H′E′F′*, the base *GHF* is equal to *ABD*; the parallelogram *GHDC*, which is equal to *ABFE*, is also equal to *ABF′E′*, and the parallelogram *GFBC*, which is equal to *ADHE*, is also equal to *ADH′E′*; therefore the three planes, which form the solid angle *G* in the prism *GHF-BCD*, are equal to the three planes, which form the solid angle *A* in the prism *ABD-H′E′F′*, each to each; they are moreover similarly disposed in the two cases; therefore these two prisms are equal, and being applied the one to the other would coincide. But one of them is symmetrical with the prism *ABD-HEF*; therefore the other *GHF-BCD* is also symmetrical with *ABD-HEF.*

LEMMA.

395. *In every prism* ABCI *the sections* NOPQR, STVXY (fig. 201), *made by parallel planes are equal polygons.* Fig. 201.

Demonstration. The sides *NO*, *ST*, are parallel, being intersections of two parallel planes by a third plane *ABGF*; these same sides *NO*, *ST*, are comprehended between the parallels *NS*, *OT*, which are sides of the prism; consequently *NO* is equal to *ST.* For a similar reason, the sides *OP*, *PQ*, *QR*, &c., of the section *NOPQR* are equal respectively to the sides *TV*, *VX*, *XY*, &c., of the section *STVXY.* Besides, the equal sides being also parallel, it follows that the angles *NOP*, *OPQ*, &c., of the first section are equal respectively to the angles *STV*, *TVX*, &c., of the second (344). Therefore the two sections *NOPQR*, *STVXY*, are equal polygons.

396. *Corollary.* Every section made in a prism parallel to its base is equal to this base.

THEOREM.

397. *The two symmetrical triangular prisms* ABD-HEF, BCD-HFG (fig. 208), *which compose the parallelopiped* AG, *are equivalent.* Fig. 208.

Demonstration. Through the vertices *B*, *F*, perpendicular to the side *BF*, suppose the planes *B a d c*, *F e h g* to pass, meeting the three other sides, *AE*, *DH*, *CG*, of the parallelopiped, the one in *a*, *d*, *c*, the other in *e*, *h*, *g*; the sections *Badc*, *Fehg*, will be equal parallelograms. They are equal, because they are made by planes, which are perpendicular to the same straight line, and consequently parallel (397); they are parallelograms, because the two opposite sides of the same section *a B*, *d c*, are the intersections of two parallel planes *ABFE*, *DCGH*, by the same plane.

For a similar reason, the figure *B a e F* is a parallelogram, as also the other lateral faces *BF g c*, *c d h g*, *a d h e*, of the solid *B a d c-F e h g*; therefore this solid is a prism (368); and this prism is a right prism, since the side *BF* is perpendicular to the plane of the base.

This being premised, if the right prism *B h* be divided by the the plane *BFHD* into two right triangular prisms *a B d-h e F*, *B d c-g F h*, we say that the oblique triangular prism *ABD-HEF* will be equivalent to the right triangular prism *a B d-h e F*.

Indeed, as the two prisms have the part *ABD h e F* common, it is necessary only to prove that the two remaining parts, namely, the solids *B a AD d*, *F e EH h*, are equivalent to each other.

Now, on account of the parallelograms *ABFE*, *a BF e*, the sides *AE*, *a e*, being each equal to its parallel *BF*, are equal to each other; if then we take away the common part *A e*, we shall have *A a* = *E e*. It may be shown, in like manner, that *D d* = *H h*.

Now, in order to apply the two solids *B a AD d*, *F e EH h*, one to the other, let the base *F e h* be placed upon its equal *B a d*; the point *e* falling upon *a*, and the point *h* upon *d*, the sides *e E*, *h H*, will fall upon *a A*, *d D*, each upon its equal, since they are perpendicular to the same plane *B a d*; consequently the two solids under consideration will coincide entirely, the one with the other; therefore the oblique prism *BAD-HFE* is equivalent to the right prism *B a d-h F e*.

It may be demonstrated, in like manner, that the oblique prism *BDC-GFH* is equivalent to the right prism *B d c-g F h*. But the the two right prisms *B a d-F c h*, *B d c-g F h*, are equal to each other, since they have the same altitude *BF*, and their bases *B a d*, *B d c*, are each half of the same parallelogram (389). Therefore the two triangular prisms *BAD-HFE*, *BDC-GFH*, equivalent to equal prisms, are equivalent to each other.

398. *Corollary.* Every triangular prism *ABD-HFE* is half of the parallelopiped *AG*, constructed upon the same solid angle *A* with the same edges *AB*, *AD*, *AE*.

THEOREM.

399. *If two parallelopipeds* AG, AL (fig. 209), *have a common base* ABCD, *and have also their superior bases comprehended in the same plane and between the same parallels* EK, HL, *these two parallelopipeds will be equivalent.* Fig. 209.

Demonstration. The proposition admits of three cases, according as *EI* is greater than *EF*, less, or equal to it; but the demonstration is the same for each; and, in the first place, we say that the triangular prism *AEI-MDH* is equal to the triangular prism *BFK-LCG*.

Indeed, since *AE* is parallel to *BF*, and *HE* to *GF*, the angle *AEI* = *BFK*, *HEI* = *GFK*, *HEA* = *GFB*. Of these six angles the three first form the solid angle *E*, and the three last the solid angle *F*; consequently, since these plane angles are equal, each to each, and similarly disposed, it follows that the solid angles *E*, *F*, are equal. Now, if the prism *AEM* be applied to the prism *BFL*, the base *AEI* being placed upon the base *BFK*, these two bases, being equal, will coincide; and, since the solid angle *E* is equal to the solid angle *F*, the side *EH* will fall upon its equal *FG*. Nothing further is necessary in order to show that the two prisms will coincide throughout; for the base *AEI* and its edge *EH* determine the prism *AEM*, as the base *BFK* and its edge *FG* determine the prism *BFL*, 388); therefore these prisms are equal.

But, if from the solid *AL* we take the prism *AEM*, there will remain the parallelopiped *AIL*; and, if from the same solid *AL* we take the prism *BFL*, there will remain the parallelopiped *AEG*; therefore the two parallelopipeds *AIL*, *AEG*, are equivalent.

THEOREM.

400. *Two parallelopipeds which have the same base and the same altitude, are equivalent.*

Demonstration. Let *ABCD* (*fig.* 210) be the common base of two parallelopipeds *AG*, *AL*; since they have the same altitude, Fig. 210.

their superior bases *EFGH*, *IKLM*, will be in the same plane. Moreover the sides *EF*, *AB*, are equal and parallel, as also *IK*, *AB*; consequently *EF* is equal and parallel to *IK*; for a similar reason, *GF* is equal and parallel to *LK*. Produce the sides *EF*, *HG*, also *LK*, *MI*, till they shall, by their intersections, form the parallelogram *NOPQ*; it is evident that this parallelogram will be equal to each of the bases *EFGH*, *IKLM*. Now, if a third parallelopiped be supposed, which, with the same inferior base *ABCD*, has for its superior base *NOPQ*, this third parallelopiped will be equivalent to the parallelopiped *AG* (399); since, the inferior base being the same, the superior bases are comprehended in the same plane and between the same parallels *GQ*, *FN*. For the same reason, this third parallelopiped will be equivalent to the parallelopiped *AL*, therefore the two parallelopipeds *AG*, *AL*, which have the same base and the same altitude, are equivalent.

THEOREM.

401. *Every parallelopiped may be changed into an equivalent rectangular parallelopiped having the same altitude and an equivalent base.*

Fig. 210. *Demonstration.* Let *AG* (*fig.* 210) be the proposed parallelopiped; from the points *A*, *B*, *C*, *D*, draw *AI*, *BK*, *CL*, *DM*, perpendicular to the plane of the base, and we shall thus have the parallelopiped *AL* equivalent to the parallelopiped *AG*, and of which the lateral faces *AK*, *BL*, &c., will be rectangles. If then the base *ABCD* is a rectangle, *AL* will be the rectangular parallelopiped equivalent to the proposed parallelopiped *AG*.

Fig. 211. But, if *ABCD* (*fig.* 211) is not a rectangle, draw *AO*, *BN*, each perpendicular to *CD*, also *OQ*, *NP*, each perpendicular to the base, and we shall have the solid *ABNO-IKPQ*, which will be a rectangular parallelopiped. Indeed the base *ABNO* and the opposite base *IKPQ* are, by construction, rectangles; the lateral faces are also rectangles, since the edges *AI*, *OQ*, &c., are each perpendicular to the plane of the base; therefore the solid *AP* is a rectangular parallelopiped. But the two parallelopipeds *AP*, *AL*, may be considered as having the same base *ABKI*, and the same altitude *AO*; consequently they are equivalent; therefore the parallelopiped *AG* (*fig.* 210, 211), which was first changed into an equivalent parallelopiped *AL*, is now changed Fig. 210, 211.

into an equivalent rectangular parallelopiped *AP*, which has the same altitude *AI*, and of which the base *ABNO* is equivalent to the base *ABCD*.

THEOREM.

402. *Two rectangular parallelopipeds* AG, AL (fig. 212), *which have the same base* ABCD, *are to each other as their altitudes*, AE, AI. Fig. 212.

Demonstration. Let us suppose, in the first place, that the altitudes *AE*, *AI*, are to each other as two entire numbers, as 15 to 8, for example, *AE* may be divided into 15 equal parts of which *AI* will contain 8, and through the points of division x, y, z, &c., planes may be drawn parallel to the base. These planes will divide the solid *AG* into 15 partial parallelopipeds, which will be equal to each other, having equal bases and equal altitudes; we say equal bases, because every section of a prism *MIKL*, parallel to the base, is equal to this base (395), and equal altitudes, because the altitudes are the divisions themselves Ax, xy, yz, &c. Now, of these 15 equal parallelopipeds 8 are contained in *AL*; therefore the solid *AG* is to the solid *AL* as 15 is to 8, or in general as the altitude *AE* is to the altitude *AI*.

Again, if the ratio of *AE* to *AI* cannot be expressed in numbers, we say still, that the proportion is not the less true, namely,

$$\text{solid } AG : \text{solid } AL :: AE : AI.$$

For, if this proportion does not hold, let us suppose that

$$\text{solid } AG : \text{solid } AL :: AE : AO.$$

Divide *AE* into equal parts, each of which shall be less than *IO*; there will be at least one point of division m between I and O. Let P be the parallelopiped which has for its base *ABCD* and for its altitude Am; since the altitudes *AE*, Am, are to each other as two entire numbers, we shall have

$$\text{solid } AG : P :: AE : Am.$$

But, by hypothesis,

$$\text{solid } AG : \text{solid } AL :: AE : AO,$$

whence

$$\text{solid } AL : P :: AO : Am.$$

But *AO* is greater than Am; it is necessary then, in order that this proportion may take place, that the solid *AL* should be greater than P; on the contrary it is less; consequently it is impossible that the fourth term of the proportion

$$\text{solid } AG : \text{solid } AL :: AE : x$$

should be a line greater than AI. By similar reasoning it may be shown that the fourth term cannot be less than AI; it is then equal to AI; therefore rectangular parallelopipeds of the same base are to each other as their altitudes.

THEOREM.

Fig. 213. 403. *Two rectangular parallelopipeds* AG, AK (fig. 213), *which have the same altitude* AE, *are to each other as their bases* ABCD, AMNO.

Demonstration. Having placed the two solids the one by the side of the other, as represented in the figure, produce the plane $ONKL$, till it meet the plane $DCGH$ in PQ, and a third parallelopiped AQ will be obtained, which may be compared with each of the parallelopipeds AG, AK. The two solids AG, AQ, having the same base $AEHD$ are to each other as their altitudes AB, AO; also the two solids AQ, AK, having the same base $AOLE$, are to each other as their altitudes AD, AM. Thus we have the two proportions

$$\text{solid } AG : \text{solid } AQ :: AB : AO,$$

$$\text{solid } AQ : \text{solid } AK :: AD : AM.$$

Multiplying the two proportions in order and omitting in the result the common multiplier *solid* AQ we shall have

$$\text{solid } AG : \text{solid } AK :: AB \times AD : AO \times AM.$$

But $AB \times AD$ represents the base $ABCD$ and $AO \times AM$ represents the base $AMNO$; therefore two rectangular parallelopipeds of the same altitude are to each other as their bases.

THEOREM.

404. *Any two rectangular parallelopipeds are to each other as the products of their bases by their altitudes, or as the products of their three dimensions.*

Fig. 213. *Demonstration.* Having placed the two solids AG, AZ (*fig.* 213), in such a manner that their surfaces may have a common angle BAE, produce the planes necessary to form the third parallelopiped AK of the same altitude with the parallelopiped AG, we shall have, by the preceding proposition,

$$\text{solid } AG : \text{solid } AK :: ABCD : AMNO.$$

But the two parallelopipeds AK, AZ, which have the same base $AMNO$, are to each other as their altitudes AE, AX; thus we have $\text{solid } AK : \text{solid } AZ :: AE : AX.$

Multiplying these two proportions in order and omitting in the result the common multiplier *solid AK*, we obtain

$$\text{solid } AG : \text{solid } AZ :: ABCD \times AE : AMNO \times AX.$$

In the place of the bases $ABCD$, $AMNO$, we can substitute $AB \times AD$, $AO \times AM$, which will give

$$\text{solid } AG : \text{solid } AZ :: AB \times AD \times AE : AO \times AM \times AX.$$

Therefore any two rectangular parallelopipeds are to each other as the products of their bases by their altitudes, or as the products of their three dimensions.

405. *Scholium.* Hence we may take for the measure of a rectangular parallelopiped the product of its base by its altitude, or the product of its three dimensions. It is on this principle that we estimate all other solids.

In order to understand this measure it is necessary to recollect that by the product of two or several lines is meant the product of the numbers which represent these lines, and these numbers depend upon the linear unit, which may be taken at pleasure; the product therefore of the three dimensions of a parallelopiped is a number which of itself has no meaning, and which would be different according as one or another linear unit is used. But if, in like manner, the three dimensions of another parallelopiped are multiplied together, by estimating them according to the same linear unit, the two products would be to each other as the two parallelopipeds and would give an idea of their relative magnitude.

The magnitude of a solid, its volume, or its extension, constitutes what is called its *solidity*; and the word *solidity*† is employed particularly to denote the measure of a solid; thus we say that the solidity of a rectangular parallelopiped is equal to the product of its base by its altitude, or the product of its three dimensions.

The three dimensions of a cube being equal to each other, if the side is 1, the solidity will be 1 × 1 × 1, or 1; if the side is

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† *Content* is often employed by English writers to denote both solid and superficial measures. The word *solidity*, though most commonly used, is exceptionable, as it is likely to suggest to the mind of the student the idea of resistance. The term *volume* has been adopted by some as preferable to solidity.

2, the solidity will be 2 × 2 × 2, or 8; if the side is 3, the solidity will be 3 × 3 × 3, or 27, and so on; thus, the sides of cubes being as the numbers 1, 2, 3, &c., the cubes themselves, or their solidities, are as the numbers 1, 8, 27, &c. Hence the origin of what in arithmetic is called the *cube* of a number; it is the product arising from three factors, which are each equal to this number.

If it were proposed to make a cube double of a given cube, it would be necessary that the side of the cube sought should be to the side of the given cube as the cube root of 2 is to 1. Now it is easy to find, by a geometrical construction, the square root of 2; but we cannot, in this way, find the cube root of this number, at least by the simple operations of elementary geometry, which consist in employing only straight lines, two points of which are known, and circles whose centres and radii are determined.

On account of this difficulty, the problem of the *duplication of the cube* was celebrated among the ancient geometers, as also that of the *trisection of an angle*, which is nearly of the same character. But the solutions, of which problems of this kind are susceptible, have long been known; and, although less simple than the constructions of elementary geometry, they are not less exact or less rigorous.

THEOREM.

406. *The solidity of a parallelopiped, and in general of any prism whatever, is equal to the product of its base by its altitude.*

Demonstration. 1. A parallelopiped of whatever kind is equivalent to a rectangular parallelopiped having the same altitude and an equivalent base (401). But the solidity of this last is equal to the product of its base by its altitude (405); therefore the solidity of the first is also equal to the product of its base by its altitude.

2. Every triangular prism is half of a parallelopiped, so constructed as to have the same altitude and a base twice as great (397). Now the solidity of this last is equal to the product of its base by its altitude (405); therefore the solidity of the triangular prism is equal to the product of its base, half of that of the parallelopiped, by its altitude.

3. A prism of whatever kind may be divided into as many triangular prisms of the same altitude, as there are triangles in the polygon taken for a base. But the solidity of each triangular prism is equal to the product of its base by its altitude; and, since the altitude is the same in each, it follows that the sum of all the partial prisms is equal to the sum of all the triangles, taken for bases, multiplied by the common altitude. Therefore the solidity of a prism of whatever kind is equal to the product of its base by its altitude.

407. *Corollary.* If we compare two prisms, which have the same altitude, the products of the bases by the altitudes will be as the bases; therefore *two prisms of the same altitude are to each other as their bases;* for a similar reason, *two prisms of the same base are to each other as their altitudes.*

LEMMA.

408. *If a pyramid* S-ABCDE (fig. 214) *is cut by a plane* a b d, Fig. 214.
parallel to the base,

1. *The sides* SA, SB, SC, ... *and the altitude* SO, *will be divided proportionally in* a, b, c, ... *and* o;

2. *The section* a b c d e *will be a polygon similar to the base* ABCDE.

Demonstration. The planes ABC, abc, being parallel, their intersections AB, ab, by a third plane SAB, will be parallel (340); consequently the triangles SAB, Sab, are similar, and

$$SA : Sa :: SB : Sb;$$

in like manner $SB : Sb :: SC : Sc$,

and so on; therefore the sides SA, SB, SC, &c., are cut proportionally at a, b, c, &c. The altitude SO is cut in the same proportion at the point O; for BO and bo are parallel (340), and consequently

$$SO : So :: SB : Sb \quad (196).$$

2. Since ab is parallel to AB, bc to BC, cd to CD, &c., the angle $abc = ABC$, the angle $bcd = BCD$, and so on. Moreover, on account of the similar triangles SAB, Sab,

$$AB : ab :: SB : Sb;$$

and, on account of the similar triangles SBC, Sbc,

$$SB : Sb :: BC : bc;$$

whence $AB : ab :: BC : bc$;

in like manner, $BC : bc :: CD : cd$,
and so on. Therefore the polygons $ABCDE$, $abcde$, have their angles equal, each to each, and their homologous sides proportional; that is, they are similar.

409. *Corollary.* Let $S\text{-}ABCDE$, $S\text{-}XYZ$, be two pyramids that have a common vertex, and whose altitudes are the same, or whose bases are situated in the same plane; if these pyramids be cut by a plane parallel to their bases, *the sections* a b c d e, x y z, thus formed, *will be to each other as the bases* ABCDE, XYZ.

For, the polygons $ABCDE$, $abcde$, being similar, their surfaces are as the squares of their homologous sides AB, ab; but

$$AB : ab :: SA : Sa,$$

consequently $ABCDE : abcde :: \overline{SA}^2 : \overline{Sa}^2$.
For the same reason,

$$XYZ : xyz :: \overline{SX}^2 : \overline{Sx}^2.$$

But, since $abcde$, xyz, are in the same plane,

$$SA : Sa :: SX : Sx,$$

whence $ABCDE : abcde :: XYZ : xyz$;
therefore the sections $abcde$, xyz, are to each other as their bases $ABCDE$, XYZ.

LEMMA.

Fig. 215. 410. *Let* S-ABC (fig. 215), *be a triangular pyramid, of which* S *is the vertex and* ABC *the base; if the sides* SA, SB, SC, AB, AC, BC, *be bisected at the points* D, E, F, G, H, I, *and through these points the straight lines* DE, EF, DF, EG, FH, EI, GI, GH, *be drawn; we say that the pyramid* S-ABC *may be considered as composed of two prisms* AGH-FDE, EGI-CFH, *equivalent to each other, and two equal pyramids* S-DEF, E-GBI.

Demonstration. It follows from the construction, that ED is parallel to BA, and GE to AS (199); hence the figure $ADEG$ is a parallelogram. For the same reason, the figure $ADFH$ is also a parallelogram; consequently the straight lines AD, GE, HF, are equal and parallel; therefore the solid $AGH\text{-}FDE$ is a prism (346).

It may be shown, in like manner, that the two figures $EFCI$, $CIGH$, are parallelograms, and that thus the straight lines EF, IC, GH, are equal and parallel; therefore the solid $EGI\text{-}CFH$ is also a prism. Now we say that these two triangular prisms are equivalent to each other.

Indeed, if upon the edges *GI*, *GE*, *GH*, the parallelopiped *GX* be formed, the triangular prism *EGI-CFH* will be half of this parallelopiped (397); on the other hand, the prism *AGH-FDE* is also equal to half of the parallelopiped *GX* (406), since they have the same altitude, and the triangle *AGH*, the base of the prism, is half of the parallelogram *GICH* (168), the base of the parallelopiped. Therefore the two prisms *EGI-CFH*, *AGH-FDE*, are equivalent to each other.

These two prisms being taken from the pyramid *S-ABC* there will remain only the two pyramids *S-DEF*, *E-GBI*; now we say that these two pyramids are equal to each other.

Indeed, since the following sides are equal, namely, $BE = SE$, $BG = AG = DE$, $EG = AD = SD$, the triangle *BEG* is equal to the triangle *ESD* (43). For a similar reason, the triangle *BEI* is equal to the triangle *ESF*; moreover the mutual inclination of the two planes *BEG*, *BEI*, is the same as that of the two planes *ESD*, *ESF*, since *BEG*, *ESD*, are in the same plane, and *BEI*, *ESF*, are also in the same plane. If then, in order to apply the one pyramid to the other, we place the triangle *EBG* upon its equal *EDS*, the plane *BEI* must fall upon the plane *ESF*; and, since the triangles are equal and similarly disposed, the point *I* will fall upon *F*, and the two pyramids will coincide throughout (384).

Therefore the entire pyramid *S-ABC* is composed of two triangular prisms *AGF*, *GIF*, equivalent to each other, and two equal pyramids *S-DEF*, *E-GBI*.

411. *Corollary* 1. From the vertex *S* let fall upon the plane *ABC* the perpendicular *SO*, and let *P* be the point, where this perpendicular meets the plane *DEF*, parallel to *ABC*; since $SD = \frac{1}{2} SA$, we have $SP = \frac{1}{2} SO$ (408), and the triangle $DEF = \frac{1}{4}$ triangle *ABC* (218); consequently the solidity of the prism

$$AGH\text{-}FDE = \tfrac{1}{4} ABC \times \tfrac{1}{2} SO;$$

and the solidity of the two prisms *AGH-FDE*, *EGI-CFH*, taken together, is equal $\frac{1}{4} ABC \times SO$. These two prisms are less than the pyramid *S-ABC*, since they are contained in it; therefore *the solidity of a triangular pyramid is greater than the fourth part of the product of its base by its altitude.*

412. *Corollary* II. If we join *DG*, *DH*, we shall have a new pyramid *D-AGH* equal to the pyramid *S-DEF*; for the base *DEF* may be placed upon its equal *AGH*, and then, the angles *SDE*, *SDF*, being equal to the angles *DAG*, *DAH*, it is manifest that *DS* will fall upon *AD* (364), and the vertex *S* upon the vertex *D*. Now the pyramid *D-AGH* is less than the prism *AGH-FDE* since it is contained in it; therefore each of the pyramids *S-DEF*, *E-GBI*, is less than the prism *AGH-FDE*; therefore the pyramid *S-ABC*, which is composed of two pyramids and two prisms, is less than four of these same prisms. But the solidity of one of these prisms $= \frac{1}{8} ABC \times SO$, and its quadruple $= \frac{1}{2} ABC \times SO$; hence *the solidity of any triangular pyramid is less than half of the product of its base by its altitude.*

THEOREM.

413. *The solidity of a triangular pyramid is equal to a third of the product of its base by its altitude.*

Fig. 215. *Demonstration.* Let *S-ABC* (*fig.* 215) be any triangular pyramid, *ABC* its base, *SO* its altitude; we say that the solidity of the pyramid *S-ABC* is equal to a third part of the product of the surface *ABC* by the altitude *SO*, so that

$$S\text{-}ABC = \tfrac{1}{3} ABC \times SO, \text{ or } = SO \times \tfrac{1}{3} ABC.$$

If this proposition be denied, the solidity *S-ABC* must be equal to the product of *SO* by a surface either greater or less than $\frac{1}{3} ABC$.

1°. Let this quantity be greater, so that we shall have

$$S\text{-}ABC = SO \times (\tfrac{1}{3} ABC + M).$$

If we make the same construction as in the preceding proposition, the pyramid *S-ABC* will be divided into two equivalent prisms *AGH-FDE*, *EGI-CFH*, and two equal pyramids *S-DEF*, *E-GBI*. Now the solidity of the prism *AGH-FDE* is $DEF \times PO$, consequently we shall have the solidity of the two prisms

$$AGH\text{-}FDE + EGI\text{-}CFH = DEF \times 2PO, \text{ or } = DEF \times SO.$$

The two prisms being taken from the entire pyramid, the remainder will be equal to double of the pyramid *S-DEF*, so that we shall have

$$2S\text{-}DEF = SO \times (\tfrac{1}{3} ABC + M - DEF).$$

But, because *SA* is double of *SD*, the surface *ABC* is quadruple of *DEF* (408), and thus

$$\tfrac{1}{3}ABC - DEF = \tfrac{4}{3}DEF - DEF = \tfrac{1}{3}DEF;$$

whence

$$2S\text{-}DEF = SO \times (\tfrac{1}{3}DEF + M),$$

or, by taking the half of each,

$$S\text{-}DEF = SP \times (\tfrac{1}{3}DEF + M).$$

It appears then, that in order to obtain the solidity of the pyramid *S-DEF*, it is necessary to add to a third of the base the same surface *M*, which was added to a third of the base of the large pyramid, and to multiply the whole by the altitude *SP* of the small pyramid.

If *SD* be bisected at the point *K*, and if through this point a plane *KLM* be supposed to pass parallel to *DEF* meeting the perpendicular *SP* in *Q*; according to what has just been demonstrated, $S\text{-}KLM = SQ \times (\tfrac{1}{3}KLM + M)$.

If we proceed thus to form a series of pyramids, the sides of which decrease in the ratio of 2 to 1, and the bases in the ratio 4 to 1, we shall soon arrive at a pyramid *S-abc*, the base of which *abc* shall be less than 6*M*. Let *So* be the altitude of this last pyramid; and its solidity, deduced from that of the preceding pyramids, will be

$$So \times (\tfrac{1}{3}abc + M).$$

But $M > \tfrac{1}{6}abc$, and consequently $\tfrac{1}{3}abc + M > \tfrac{1}{2}abc$. It would follow then, that the solidity of the pyramid *S-abc* is greater than $So \times \tfrac{1}{2}abc$; which is absurd, since it was proved in the preceding proposition, corollary II, that the solidity of a triangular pyramid is always less than half of the product of its base by its altitude; therefore it is impossible that the solidity of the pyramid *S-ABC* should be greater than $SO \times \tfrac{1}{3}ABC$.

2. Let *S-ABC* be equal to $SO \times (\tfrac{1}{3}ABC - M)$; it may be shown, as in the first case, that the solidity of the pyramid *S-DEF*, the dimensions of which are less by one half, is equal to

$$SP \times (\tfrac{1}{3}DEF - M);$$

and, by continuing the series of pyramids, the sides of which decrease in the ratio of 2 to 1, until we arrive at a term *S-abc*, we shall, in like manner, have the solidity of this last equal to

$$So \times (\tfrac{1}{3}abc - M).$$

But, as the bases *ABC*, *DEF*, *KLM* ..... *abc*, form a decreasing series, each term of which is a fourth of the preceding, we shall soon arrive at a term *abc* equal to 12*M*, or which shall be comprehended between 12*M* and 3*M*; then, *M* being either equal to or greater than $\frac{1}{12}\,abc$, the quantity $\frac{1}{3}\,abc - M$ will either be equal to or less than $\frac{1}{4}\,abc$; so that we shall have the solidity of the pyramid $S\text{-}abc$ either equal to or less than

$$So \times \tfrac{1}{4}\,abc;$$

which is absurd, since, according to corollary I of the preceding proposition, the solidity of a triangular pyramid is always greater than the fourth of the product of its base by its altitude; therefore the solidity of the pyramid $S\text{-}ABC$ cannot be less than $SO \times \frac{1}{3}\,ABC$.

We conclude then, according to the enunciation of the theorem, that the solidity of the pyramid $SABC = SO \times \frac{1}{3}\,ABC$, or $= \frac{1}{3}\,ABC \times SO$.

414. *Corollary* I. Every triangular pyramid is a third of a triangular prism of the same base and same altitude; for $ABC \times SO$ is the solidity of the prism of which ABC is the base and SO the altitude.

415. *Corollary* II. Two triangular pyramids of the same altitude are to each other as their bases, and two triangular pyramids of the same base are to each other as their altitudes.

THEOREM.

Fig. 214. 416. *Every pyramid* S-ABCDE (fig. 214) *has for its measure a third of the product of its base by its altitude.*

Demonstration. If the planes SEB, SEC, be made to pass through the diagonals EB, EC, the polygonal pyramid $S\text{-}ABCDE$ will be divided into several triangular pyramids, which have all the same altitude SO. But, by the preceding theorem, these are measured by multiplying their bases ABE, BCE, CDE, each by a third of its altitude SO; consequently the sum of the triangular pyramids, or the polygonal pyramid $S\text{-}ABCDE$ will have for its measure the sum of the triangles ABE, BCE, CDE, or the polygon $ABCDE$, multiplied by $\frac{1}{3}\,SO$; therefore every pyramid has for its measure a third of the product of its base by its altitude.

417. *Corollary* I. Every pyramid is a third of a prism of the same base and same altitude.

418. *Corollary* II. Two pyramids of the same altitude are to each other as their bases, and two pyramids of the same base are to each other as their altitudes.

419. *Scholium.* The solidity of any polyedron may be estimated by decomposing it into pyramids, and this decomposition may be effected in several ways; one of the most simple is by means of planes of division passing through the vertex of the same solid angle; then we shall have as many partial pyramids, as there are faces in the polyedron excepting those which contain the solid angle from which the planes of division proceed.

THEOREM.

420. *Two symmetrical polyedrons are equivalent to each other or equal in solidity.*

Demonstration. 1. Two symmetrical triangular pyramids, as *S-ABC*, *T-ABC* (*fig.* 202), have each for its measure the product of the base *ABC* by a third of its altitude *SO* or *TO*; therefore these pyramids are equivalent. Fig. 202.

2. If we divide, in any manner, one of the symmetrical polyedrons in question into triangular pyramids, we can divide the other polyedron, in the same manner into triangular pyramids symmetrical with the former (382); but the triangular pyramids in the one case and the other being symmetrical, are equivalent, each to each; therefore the entire polyedrons are equivalent to each other or equal in solidity.

421. *Scholium.* This proposition seems to result immediately from a former (386), in which it was shown that, with respect to two symmetrical polyedrons, all the constituent parts of the one are equal respectively to those of the other; still it was necessary to demonstrate it in a rigorous manner.

THEOREM.

422. *If a pyramid is cut by a plane parallel to its base, the frustum which remains, after taking away the smaller pyramid, is equal to the sum of three pyramids, which have for their common altitude the altitude of the frustum, and whose bases are the inferior base of the frustum, its superior base, and a mean proportional between these bases.*

Demonstration. Let *S-ABCDE* (*fig.* 217) be a pyramid cut by the plane *a b d* parallel to the base; let *T-FGH* be a triangular pyramid, whose base and altitude are equal or equivalent to the base and altitude of the pyramid *S-ABCDE*. The two Fig. 217.

bases may be supposed to be situated in the same plane; and then the plane *a b d* produced, will determine in the triangular pyramid a section *f g h* situated at the same altitude above the common plane of the bases; whence it follows that the section *f g h* is to the section *a b d*, as the base *FGH* is to the base *ABD* (408); and, since the bases are equivalent, the sections will be equivalent also. Consequently the pyramids *S-a b c d e*, *T-f g h*, are equivalent, since they have the same altitude and equivalent bases. The entire pyramids *S-ABCDE*, *T-FGH*, are equivalent, for the same reason; therefore the frustums *ABD-d a b*, *FGH-h f g*, are equivalent; and consequently it will be sufficient to demonstrate the proposition enunciated, with reference merely to the case of the frustum of a triangular pyramid.

Fig. 218. Let *FGH-h f g* (*fig.* 218), be the frustum of a triangular pyramid; through the points *F*, *g*, *H*, suppose a plane *F g H*, to pass cutting off from the frustum the triangular pyramid *g-FGH*. This pyramid has for its base the inferior base *FGH* of the frustum, it has also for its altitude the altitude of the frustum, since the vertex *g* is in the plane of the superior base *f g h*.

This pyramid being cut off, there will remain the quadrangular pyramid *g-fh HF*, the vertex of which is *g* and the base *f h HF*. Through the three points *f*, *g*, *H*, suppose a plane *f g H* to pass dividing the quadrangular pyramid into two triangular pyramids *g-F f H*, *g-f h H*. This last pyramid may be considered as having for its base the superior base *g f h* of the frustum, and for its altitude the altitude of the frustum, since the vertex *H* is in the inferior base. Thus we have two of the three pyramids which compose the frustum.

It remains to consider the third pyramid *g-F f H*. Now if we draw *g K* parallel to *f F*, and suppose a new pyramid *K-F f H*, the vertex of which is *K*, and the base *F f H*; these two pyramids will have the same base *F f H*; they will have also the same altitude, since the vertices *g*, *K*, are situated in a line *g K* parallel to *F f*, and consequently parallel to the plane of the base; therefore these pyramids are equivalent. But the pyramid *K-F f H* may be considered as having its vertex in *f*, and thus it will have the same altitude as the frustum; as to its base *FHK*, we say that it is a mean proportional between the two bases *FHG*, *f h g*. Indeed the triangles *FHK*, *f h g*, have the angle $F = f$, and the side $FK = fg$, hence

$$FHK : fhg :: FK \times FH : fg \times fh :: FH : fh \quad (216).$$

Also $FHG : FHK :: FG : FK$ or fg.

But the similar triangles FHG, fhg, give

$$FG : fg :: FH : fh;$$

consequently $FHG : FHK :: FHK : fhg$;

and thus the base FHK is a mean proportional between the two bases FHG, fhg, therefore the frustum of a triangular pyramid is equal to three pyramids, which have for their common altitude the altitude of the frustum, and whose bases are the inferior base of the frustum, its superior base, and a mean proportional between these bases.

THEOREM.

423. *If a triangular prism, whose base is* ABC (fig. 216), *be cut by a plane* DEF *inclined to this base, the solid* ABC-DEF *thus formed, will be equal to the sum of the three pyramids whose vertices are* D, E, F, *and the common base* ABC. Fig. 216.

Demonstration. Through the three points *F*, *A*, *C*, suppose a plane *FAC* to pass cutting off from the truncated prism

ABC-DEF

the triangular pyramid *F-ABC*; this pyramid will have for its base *ABC* and for its vertex the point *F*.

This pyramid being cut off, there will remain the quadrangular pyramid *F-ACDE*, of which *F* is the vertex, and *ACDE* the base. Through the points *F*, *E*, *C*, suppose a plane *FEC* to pass dividing the quadrangular pyramid into two triangular pyramids *F-AEC*, *F-CDE*.

The pyramid *F-AEC*, which has for its base the triangle *AEC* and for its vertex the point *F*, is equivalent to a pyramid *B-AEC*, which has for its base *AEC*, and for its vertex the point *B*. For these two pyramids have the same base; they have also the same altitude, since the line *BF*, being parallel to each of the lines *AE*, *CD*, is parallel to their plane *AEC*; therefore the pyramid *F-AEC* is equivalent to the pyramid *B-AEC*, which may be considered as having for its base *ABC*, and for its vertex the point *E*.

The third pyramid *F-CDE*, or *E-FCD*, may be changed in the first place into *A-FCD*; for the two pyramids have the same base *FCD*; they have also the same altitude, since *AE* is parallel to the plane *FCD*; consequently the pyramid *E-FCD* is equivalent to *A-FCD*. Again, the pyramid *A-FCD*, or *F-ACD*, may be changed into *B-ACD*, for these two pyramids have the common base *ACD*; they have also the same altitude, since their vertices *F* and *B* are in a parallel to the plane of the base. Therefore the pyramid *E-FCD*, equivalent to *A-FCD*, is also equivalent to *B-ACD*. Now this last may be regarded as having for its base *ABC*, and for its vertex the point *D*.

We conclude then, that the truncated prism *ABC-DEF* is equal to the sum of three pyramids which have for their common base *ABC* and whose vertices are respectively the points *D*, *E*, *F*.

424. *Corollary.* If the edges are perpendicular to the plane of the base, they will be at the same time the altitudes of the three pyramids, which compose the truncated prism; so that the solidity of the truncated prism will be expressed by

$$\tfrac{1}{3} ABC \times AE + \tfrac{1}{3} ABC \times BF + \tfrac{1}{3} ABC \times CD,$$

or

$$\tfrac{1}{3} ABC \times (AE + BF + CD).$$

THEOREM.

425. *Two similar triangular pyramids have their homologous faces similar, and their homologous solid angles equal.*

Demonstration. The two triangular pyramids *S-ABC*, *T-DEF* (*fig.* 203), are similar, if the two triangles *SAB*, *ABC*, are similar to the two *TDE*, *DEF*, and are similarly placed (381); that is, if the angle *ABS* = *DET*, *BAS* = *EDT*, *ABC* = *DEF*, *BAC* = *EDF*, and if furthermore the inclination of the planes *SAB*, *ABC*, is equal to that of the planes *TDE*, *DEF*. This being supposed, we say that the pyramids have all their faces similar, each to each, and their homologous solid angles equal.

Fig. 203.

Take *BG* = *ED*, *BH* = *EF*, *BI* = *ET*, and join *GH*, *GI*, *IH*. The pyramid *T-DEF* is equal to the pyramid *I-GBH*; for the sides *GB*, *BH*, being equal, by construction, to the sides *DE*, *EF*, and the angle *GBH* being, by hypothesis, equal to the angle *DEF*, the triangle *GBH* is equal to *DEF* (36); therefore, in order to apply one of these pyramids to the other, we can evidently place the base *DEF* upon its equal *GBH*; then, since the plane *TDE* has the same inclination to *DEF*, as the plane *SAB* has to

ABC, it is manifest that the plane *TDE* will fall indefinitely upon the plane *SAB*. But, by hypothesis, the angle *DET* = *GBI*, consequently *ET* will fall upon its equal *BI*; and since the four points *D*, *E*, *F*, *T*, coincide with the four *G*, *B*, *H*, *I*, it follows that the pyramid *T-DEF* will coincide with the pyramid *I-GBH* (384).

Now, on account of the equal triangles *DEF*, *GBH*, the angle *BGH* = *EDF* = *BAC*; consequently *GH* is parallel to *AC*. For a similar reason *GI* is parallel to *AS*; therefore the plane *IGH* is parallel to *SAC* (344). Whence it follows that the triangle *IGH*, or its equal *TDF*, is similar to *SAC* (347), and that the triangle *IBH*, or its equal *TEF*, is similar to *SBC*; therefore the two similar triangular pyramids *S-ABC*, *T-DEF* have their four faces similar, each to each. Moreover the homologous solid angles are equal.

For, we have already placed the solid angle *E* upon its homologous angle *B*, and the same may be done with respect to the two other homologous solid angles; but it will be readily perceived that two homologous solid angles are equal, for example the angles *T* and *S*, because they are formed by three plane angles which are equal, each to each, and similarly placed.

Therefore two similar triangular pyramids have their homologous faces similar and the homologous solid angles equal.

426. *Corollary* I. The similar triangles in the two pyramids furnish the proportions

$AB : DE :: BC : EF :: AC : DF :: AS : DT : SB : TE :: SC : TF$; therefore *in similar triangular pyramids the homologous sides are proportional.*

427. *Corollary* II. Since the homologous solid angles are equal, it follows that *the inclination of any two faces of one pyramid is equal to the inclination of the two homologous faces of a similar pyramid* (359).

428. *Corollary* III. If a triangular pyramid *SABC* be cut by a plane *GIH* parallel to one of the faces *SAC*, the partial pyramid *IGBH* will be similar to the entire pyramid *SABC*. For the triangles *BGI*, *BGH*, are similar to the triangles *BAS*, *BAC*, each to each, and similarly placed; also the inclination of the two planes is the same in each; therefore the two pyramids are similar.

429. *Corollary* IV. *If any pyramid whatever* SABCDE (fig. 214) Fig. 214.

be cut by a plane a b c d e *parallel to the base, the partial pyramid* S-a b c d e *will be similar to the entire pyramid* S-ABCDE. For the bases $ABCDE$, $abcde$, are similar, and AC, ac, being joined, it has just been proved that the triangular pyramid S-ABC is similar to the pyramid S-abc; therefore the point S is determined with respect to the base ABC, as the point S is determined with respect to the base abc (382); therefore the two pyramids S-$ABCDE$, S-$abcde$, are similar.

430. *Scholium.* Instead of the five given things, required by the definition in order that two triangular pyramids may be similar, we can substitute five others, according to different combinations; and there will result as many theorems, among which may be distinguished the following; *two triangular pyramids are similar, when they have their homologous sides proportional.*

For, if we have the proportions

$$AB:DE::BC:EF::AC:DF::AS:DT::SB:TE::SC:TF$$

Fig. 203. (*fig.* 203), which contain five conditions, the triangles ABS, ABC, will be similar to DET, DEF, and the disposition of the former will be similar to that of the latter. We have also the triangle SBC similar to TEF; therefore the three plane angles, which form the solid angle B, are equal to the three plane angles which form the solid angle E, each to each; whence it follows that the inclination of the planes SAB, ABC, is equal to that of the homologous planes TDE, DEF, and that thus the two pyramids are similar.

THEOREM.

431. *Two similar polyedrons have their homologous faces similar, and their homologous solid angles equal.*

Fig. 219. *Demonstration.* Let $ABCDE$ (*fig.* 219) be the base of one polyedron; let M, N, be the vertices of two solid angles, without this base, determined by the triangular pyramids M-ABC, N-ABC, whose common base is ABC; let there be, in the other polyedron, the base $abcde$ homologous or similar to $ABCDE$, m, n, the vertices homologous to M, N, determined by the pyramids m-abc, n-abc, similar to the pyramids M-ABC, N-ABC; we say, in the first place, that the distances MN, mn, are proportional to the homologous sides AB, ab.

Indeed, the pyramids M-ABC, m-abc, being similar, the inclination of the planes MAC, BAC, is equal to that of the planes *m a c*, *b a c*; in like manner, the pyramids *N-ABC*, *n-a b c*, being similar, the inclination of the planes *NAC*, *BAC*, is equal to that of the planes *n a c*, *b a c*; consequently, if we subtract the first inclinations respectively from the second, there will remain the inclination of the planes *NAC*, *MAC*, equal to that of the planes *n a c*, *m a c*. But, because the pyramids are similar, the triangle *MAC* is similar to *m a c*, and the triangle *NAC* is similar to *n a c*; therefore the triangular pyramids *MNAC*, *m n a c*, have two faces similar each to each, similarly placed, and equally inclined to each other; consequently the two pyramids are similar (425); and their homologous sides give the proportion

$$MN : mn :: AM : am.$$

Moreover $AM : am :: AB : ab;$

therefore, $MN : mn :: AB : ab.$

Let P and p be two other homologous vertices of the same polyedrons, and we have, in like manner,

$$PN : pn :: AB : ab,$$
$$PM : pm :: AB : ab;$$

whence $MN : mn :: PN : pn :: PM : pm.$

Therefore *the triangle* PNM, *formed by joining any three vertices of one polyedron, is similar to the triangle* p n m, *formed by joining the three homologous vertices of the other polyedron.*

Furthermore, let Q, q, be two homologous vertices, and the triangle *PQN* will be similar to *p q n*. We say also, that the inclination of the planes *PQN*, *PMN*, is equal to that of the planes *p q n*, *p m n*.

For, if we join *QM* and *q m*, we shall always have the triangle *QNM* similar to *q n m*, and consequently the angle *QNM* equal to *q n m*. Suppose at *N* a solid angle formed by the three plane angles *QNM*, *QNP*, *PNM*, and at *n* a solid angle formed by the plane angles *q n m*, *q n p*, *p n m*; since these plane angles are equal, each to each, it follows that the solid angles are equal. Whence the inclination of the two planes *PNQ*, *PNM*, is equal to that of the homologous planes *p n q*, *p n m* (359); therefore, if the two triangles *PNQ*, *PNM*, be in the same plane, in which case we should have the angle $QNM = QNP + PNM$, we should have, in like manner, the angle $qnm = qnp + pnm$, and the two triangles *q n p*, *p n m*, would also be in the same plane.

All that has now been demonstrated takes place, whatever be the angles M, N, P, Q, compared with the homologous angles m, n, p, q.

Let us suppose now that the surface of one of the polyedrons is divided into triangles ABC, ACD, MNP, NPQ, &c., we see that the surface of the other polyedron will contain an equal number of triangles, abc, acd, mnp, npq, &c., similar to the former and similarly placed; and if several triangles, as MPN, NPQ, &c. belong to the same face, and are in the same plane, the homologous triangles mpn, npq, &c., will likewise be in the same plane. Therefore each polygonal face in the one polyedron will correspond to a similar polygonal face in the other; and consequently the two polyedrons will be comprehended under the same number of similar and similarly disposed planes. We say moreover, that the solid angles will be equal.

For, if the solid angle N, for example, is formed by the plane angles QNP, PNM, MNR, QNR, the homologous solid angle n will be formed by the plane angles qnp, pnm, mnr, qnr. Now the former plane angles are equal to the latter, each to each, and the inclination of any two adjacent planes, is equal to that of their homologous planes; therefore the two solid angles are equal, since they would coincide upon being applied.

We conclude then, that two similar polyedrons have their homologous faces similar, and their homologous solid angles equal.

432. *Corollary.* It follows, from the preceding demonstration, that if with four vertices of a polyedron we form a triangular pyramid, and also another with the four homologous vertices of a similar polyedron, these two pyramids will be similar; for they will have their homologous sides proportional (430).

It will be perceived, at the same time, that the homologous diagonals (157), AN, an, for example, are to each other as two homologous sides AB, ab.

THEOREM.

433. *Two similar polyedrons may be divided into the same number of triangular pyramids similar, each to each, and similarly placed.*

Demonstration. We have seen that the surfaces of two similar polyedrons may be divided into the same number of triangles, that are similar, each to each, and similarly placed. Let us consider all the triangles of one of the polyedrons, except those which form the solid angle *A*, as the bases of so many triangular pyramids having their vertices in *A*; these pyramids taken together will compose the polyedron. Let us divide likewise the other polyedron into pyramids having for their common vertex that of the angle *a*, homologous to *A*; it is evident that the pyramid, which connects four vertices of one polyedron, will be similar to the pyramid which connects the four homologous vertices of the other polyedron; therefore two similar polyedrons, &c.

THEOREM.

434. *Two similar pyramids are to each other as the cubes of their homologous sides.*

Demonstration. Two pyramids being similar, the less may be placed in the greater so that they shall have the angle S (*fig.* 214) common. Then the bases $ABCDE$, $a\,b\,c\,d\,e$, will be parallel; for, since the homologous faces are similar (423), the angle Fig. 214.

$$S\,a\,b = SAB,$$

as also $S\,b\,c = SBC$; therefore the plane $a\,b\,c$ is parallel to the plane ABC (344). This being premised, let fall the perpendicular SO from the vertex S upon the plane ABC, and let o be the point, where this perpendicular meets the plane $a\,b\,c$; we shall have, according to what has already been demonstrated (406), $SO : S\,o :: SA : S\,a :: AB : a\,b$,

and consequently

$$\tfrac{1}{3}\,SO : \tfrac{1}{3}\,S\,o :: AB : a\,b.$$

But the bases $ABCDE$, $a\,b\,c\,d\,e$, being similar figures,

$$ABCDE : a\,b\,c\,d\,e :: \overline{AB}^2 : \overline{a\,b}^2 \quad (221).$$

Multiplying the two proportions in order we shall have

$$ABCDE \times \tfrac{1}{3}\,SO : a\,b\,c\,d\,e \times \tfrac{1}{3}\,S\,o :: \overline{AB}^3 : \overline{a\,b}^3;$$

but $ABCDE \times \frac{1}{3}\,SO$ is the solidity of the pyramid $SABCDE$ (413), and $a\,b\,c\,d\,e \times \frac{1}{3}\,S\,o$ is the solidity of the pyramid $S\,a\,b\,c\,d\,e$; therefore two similar pyramids are to each other as the cubes of their homologous sides.

THEOREM.

435. *Two similar polyedrons are to each other as the cubes of their homologous sides.*

Demonstration. Two similar polyedrons may be divided into the same number of triangular pyramids that are similar, each to each (433). Now the two similar pyramids $APNM$, $apnm$, Fig 219. (*fig.* 219), are to each other as the cubes of their homologous sides AM, am, or as the cubes of the homologous sides AB, ab, (434). The same ratio may be shown to exist between any two other homologous pyramids; therefore the sum of all the pyramids, which compose the one polyedron, or the polyedron itself, is to the other polyedron, as the cube of any one of the sides of the first, is to the cube of the homologous side of the second.

General Scholium.

436. We can express in algebraic language, that is, in a manner the most concise, a recapitulation of the principal propositions of this section relating to the solidity or content of polyedrons.

Let B be the base of a prism, H its altitude; the solidity of the prism will be $B \times H$, or BH.

Let B be the base of a pyramid, H its altitude; the solidity of the pyramid will be $B \times \frac{1}{3} H$, or $H \times \frac{1}{3} B$, or $\frac{1}{3} BH$.

Let H be the altitude of the frustum of a pyramid and let A, B, be the bases; then $\sqrt{AB}$ will be the mean proportion between them, and the solidity of the frustum will be

$$\tfrac{1}{3} H \times (A + B + \sqrt{AB}).$$

Let B be the base of a truncated triangular prism, H, H', H'', the altitudes of the three superior vertices, the solidity of the truncated prism will be $\frac{1}{3} B \times (H + H' + H'')$.

Lastly, let P, p, be the solidities of two similar polyedrons, A and a, two homologous sides, or diagonals of the polyedrons, we shall have

$$P : p :: A^3 : a^3.$$

SECTION THIRD.

Of the Sphere.

DEFINITIONS.

437. A *sphere* is a solid terminated by a curved surface all the points of which are equally distant from a point within called the *centre.*

The sphere may be conceived to be generated by the revolution of a semicircle *DAE* (*fig.* 220) about its diameter *DE*; for the surface thus described by the curve *DAE* will have all its points equally distant from the centre *C*. Fig. 220.

438. The *radius of a sphere* is a straight line drawn from the centre to a point in the surface; the *diameter* or *axis*, is a line passing through the centre and terminated each way by the surface.

All radii of the same sphere are equal; the diameters also are equal, and each double of the radius.

439. It will be demonstrated art. 452, that every section of a sphere made by a plane is a circle. This being supposed, we call a *great circle* the section made by a plane which passes through the centre, and a *small circle* the section made by a plane which does not pass through the centre.

440. A *plane* is a *tangent* to a sphere, when it has one point only in common with the surface of the sphere.

441. The *pole of a circle* of the sphere is a point in the surface of the sphere equally distant from every point in the circumference of the circle. It will be shown art. 464, that every circle great or small has two poles.

442. A *spherical triangle* is a part of the surface of a sphere comprehended by three arcs of great circles.

These arcs, which are called the *sides* of the triangle, are always supposed to be smaller each than a semicircumference. The angles which their planes make with each other are the angles of the triangle.

443. A spherical triangle takes the name of *right-angled*, *isosceles* and *equilateral*, like a plane triangle, and under the same circumstances.

444. A *spherical polygon* is a part of the surface of a sphere terminated by several arcs of great circles.

455. A *lunary surface* is the part of the surface of a sphere comprehended between two semicircumferences of great circles, which terminate in a common diameter.

446. We shall call a *spherical wedge* the part of a sphere comprehended between the halves of two great circles, and to which the lunary surface answers as a base.

447. A *spherical pyramid* is the part of a sphere comprehended between the planes of a solid angle whose vertex is at the centre. The *base* of the pyramid is the spherical polygon intercepted by these planes.

448. A *zone* is the part of the surface of a sphere comprehended between two parallel planes, which are its *bases*. One of these planes may be a tangent to the sphere, in which case the zone has only one base.

449. A *spherical segment* is the portion of a sphere comprehended between two parallel planes which are its bases. One of these planes may be a tangent to the sphere, in which case the spherical segment has only one base.

450. *The altitude of a zone* or *of a segment* is the distance between the parallel planes which are the bases of the zone or segment.

Fig. 220. 451. While the semicircle DAE (*fig.* 220), turning about the diameter DE, describes a sphere, every circular sector as DCF, or FCH, describes a solid which is called a *spherical sector.*

THEOREM.

452. *Every section of a sphere made by a plane is a circle.*

Fig. 221. *Demonstration.* Let AMB (*fig.* 221) be a section, made by a plane, of the sphere of which C is the centre. From the point C draw CO perpendicular to the plane AMB, and different oblique lines CM, CM, to different points of the curve AMB which terminates the section.

The oblique lines CM, CM, CB, are equal, since they are radii of the sphere; consequently they are at equal distances from the perpendicular CO (329); whence all the lines OM, OM, OB, are equal; therefore the section AMB is a circle of which the point O is the centre.

453. *Corollary* I. If the cutting plane pass through the centre of the sphere, the radius of the section will be the radius of the sphere; therefore all great circles are equal to each other.

454. *Corollary* II. Two great circles always bisect each other; for the common intersection, passing through the centre, is a diameter.

455. *Corollary* III. Every great circle bisects the sphere and its surface; for, if having separated the two hemispheres from each other, we apply the base of the one to that of the other, turning the convexities the same way, the two surfaces will coincide with each other; if they did not, there would be points in these surfaces unequally distant from the centre.

456. *Corollary* IV. The centre of a small circle and that of the sphere are in the same straight line perpendicular to the plane of the small circle.

457. *Corollary* V. Small circles are less according to their distance from the centre of the sphere; for, the greater the distance *CO*, the smaller the chord *AB*, the diameter of the small circle *AMB*.

458. *Corollary* VI. Through two given points on the surface of a sphere an arc of a great circle may be described; for the two given points and the centre of the sphere determine the position of a plane. If, however, the two given points be the extremities of a diameter, these two points and the centre would be in a straight line, and any number of great circles might be made to pass through the two given points.

THEOREM.

459. *In any spherical triangle* ABC (fig. 222) *either side is less than the sum of the other two.* Fig. 222.

Demonstration. Let *O* be the centre of the sphere, and let the radii *OA*, *OB*, *OC*, be drawn. If the planes *AOB*, *AOC*, *COB*, be supposed, these planes will form at the point *O* a solid angle, and the angles *AOB*, *AOC*, *COB*, will have for their measure the sides *AB*, *AC*, *BC*, of the spherical triangle *ABC* (123). But each of the three plane angles, which form the solid angle, is less than the sum of the two others (356); therefore either side of the triangle *ABC* is less than the sum of the other two.

THEOREM.

460. *The shortest way from one point to another on the surface of a sphere is the arc of a great circle which joins the two given points.*

Fig. 223. *Demonstration.* Let ANB (*fig.* 223) be the arc of a great circle which joins the two given points A and B, and let there be without this arc, if it be possible, a point M of the shortest line between A and B. Through the point M draw the arcs of great circles MA, MB, and take $BN = MB$.

According to the preceding theorem the arc ANB is less than $AM + MB$; taking from one BN, and from the other its equal BM, we shall have $AN < AM$. Now the distance from B to M, whether it be the same as the arc BM, or any other line, is equal to the distance from B to N; for, by supposing the plane of the great circle BM to turn about the diameter passing through B, the point M may be reduced to the point N, and then the shortest line from M to B, whatever it may be, is the same as that from N to B; consequently the two ways from A to B, the one through M and the other through N, have the part from M to B equal to that from N to B. But the first way is, by hypothesis, the shortest; consequently the distance from A to M is less than the distance from A to N, which is absurd, since the arc AM is greater than AN; whence no point of the shortest line between A and B can be without the arc ANB; therefore this line is itself the shortest that can be drawn between its extremities.

THEOREM.

461. *The sum of the three sides of a spherical triangle is less than the circumference of a great circle.*

Fig. 224. *Demonstration.* Let ABC (*fig.* 224) be any spherical triangle; produce the sides AB, AC, till they meet again in D. The arcs ABD, ACD, will be the semicircumferences of great circles, since two great circles always bisect each other (454); but in the triangle BCD the side $BC < BD + CD$ (459); adding to each $AB + AC$ we shall have $AB + AC + BC < ABD + ACD$, that is, less than the circumference of a great circle.

THEOREM.

462. *The sum of the sides of any spherical polygon is less than the circumference of a great circle.*

Demonstration. Let there be, for example, the pentagon *ABCDE* (*fig.* 225); produce the sides *AB*, *DC*, till they meet in *F*; since *BC* is less than *BF* + *CF*, the perimeter of the pentagon *ABCDE* is less than that of the quadrilateral *AEDF*. Again produce the sides *AE*, *FD*, till they meet in *G*, and we shall have $ED < EG + GD$; consequently the perimeter of the quadrilateral *AEDF* is less than that of the triangle *AFG*; but this last is less than the circumference of a great circle (461); therefore for a still stronger reason the perimeter of the polygon *ABCDE* is less than this same circumference. Fig. 225.

463. *Scholium.* This proposition is essentially the same as that of art. 357; for if *O* be the centre of the sphere, we can suppose at the point *O* a solid angle formed by the plane angles *AOB*, *BOC*, *COD*, &c., and the sum of these angles must be less than four right angles, which does not differ from the proposition enunciated above; but the demonstration just given is different from that of art. 357; it is supposed in each that the polygon *ABCDE* is convex, or that no one of the sides produced would cut the figure.

THEOREM.

464. *If the diameter* DE (fig. 220) *be drawn perpendicular to the plane of the great circle* AMB, *the extremities* D *and* E *of this diameter will be the poles of the circle* AMB, *and of every small circle* FNG *which is parallel to it.* Fig. 220.

Demonstration. *DC*, being perpendicular to the plane *AMB*, is perpendicular to all the straight lines *CA*, *CM*, *CB*, &c., drawn through its foot in this plane (325); consequently all the arcs *DA*, *DM*, *DB*, &c., are quarters of a circumference. The same may be shown with respect to the arcs *EA*, *EM*, *EB*, &c., whence the points *D*, *E*, are each equally distant from all the points in the circumference of the circle *AMB*; therefore they are the poles of this circle (441).

Again, the radius *DC*, perpendicular to the plane *AMB*, is perpendicular to its parallel *FNG*; consequently it passes through the centre *O* of the circle *FNG* (456); whence, if *DF*, *DN*, *DG*, be drawn, these oblique lines will be equally distant from the perpendicular *DO*, and will be equal (329). But, the chords being equal, the arcs are equal; consequently all the arcs *DF*, *DN*, *DG*, &c., are equal; therefore the point *D* is the pole of the small circle, *FNG*, and for the same reason the point *E* is the other pole.

465. *Corollary* I. Every arc *DM*, drawn from a point in the arc of a great circle *AMB* to its pole, is the fourth part of the circumference, which for the sake of conciseness we shall call a *quadrant;* and this quadrant at the same time makes a right angle with the arc *AM*. For the line *DC* being perpendicular to the plane *AMC*, every plane *DMC*, which passes through the line *DC*, is perpendicular to the plane *AMC* (351); therefore the angle of these planes, or, according to the definition art. 442, the angle *AMD* is a right angle.

466. *Corollary* II. In order to find the pole of a given arc *AM*, draw the indefinite arc *MD* perpendicular to *AM*, take *MD* equal to a quadrant, and the point *D* will be one of the poles of the arc *MD*; or rather, draw to the two points *A*, *M*, the arcs *AD*, *MD*, perpendicular each to *AM*, the point of meeting *D* of these two arcs will be the pole required.

467. *Corollary* III. Conversely, if the distance of the point *D* from each of the points *A*, *M*, is equal to a quadrant, we say that the point *D* will be the pole of the arc *AM*, and that, at the same time, the angles *DAM*, *AMD*, will be right angles.

For, let C be the centre of the sphere, and let the radii *CA*, *CD*, *CM*, be drawn. Since the angles *ACD*, *MCD*, are right angles, the line *CD* is perpendicular to the two straight lines *CA*, *CM*; whence it is perpendicular to their plane (325); therefore the point *D* is the pole of the arc *AM*; and consequently the angles *DAM*, *AMD*, are right angles.

468. *Scholium.* By means of poles, arcs may be traced upon the surface of a sphere as easily as upon a plane surface. We see, for example, that by turning the arc *DF*, or any other line of the same extent about the point *D*, the extremity *F* will describe the small circle *FNG*; and, by turning the quadrant *DFA* about the point *D*, the extremity *A* will describe the arc of a great circle *AM*.

If the arc *AM* is to be produced, or if only the points *A*, *M*, are given, through which this arc is to pass, we determine, in the first place, the pole *D* by the intersection of two arcs described from the points *A*, *M*, as centres, with an extent equal to a quadrant. The pole *D* being found, we describe from the point *D*, as a centre, and with the same extent, the arc *AM* or the continuation of it.

If it is required to let fall a perpendicular from a given point *P* upon a given arc *AM*, we produce this arc to *S*, so that the distance *PS* shall be equal to a quadrant; then from the pole *S* and with the distance *PS* we describe the arc *PM*, which will be the perpendicular arc required.

THEOREM.

469. *Every plane perpendicular to the radius at its extremity is a tangent to the sphere.*

Demonstration. Let *FAG* (*fig.* 226) be a plane perpendicular to the radius *AO* at its extremity; if we take any point *M* in this plane and join *OM*, *AM*, the angle *OAM* will be a right angle, and thus the distance *OM* will be greater than *OA*; consequently the point *M* is without the sphere; and, as the same might be shown with respect to every other point of the plane *FAG*, it follows that this plane has only the point *A* in common with the surface of the sphere; therefore it is a tangent to this surface (440). Fig. 226.

470. *Scholium.* It may be shown, in like manner, that two spheres have only one point common, and are consequently tangents to each other, when the distance of their centres is equal to the sum or the difference of their radii; in this case, the centres and the point of contact are in the same straight line.

THEOREM.

471. *The angle* BAC (fig. 226), *which two arcs of great circles make with each other, is equal to the angle* FAG *formed by the tangents of these arcs at the point* A; *it has also for its measure the arc* DE, *described from the point* A *as a pole, and comprehended between the sides* AB, AC, *produced if necessary.* Fig. 226.

Demonstration. For the tangent *AF*, drawn in the plane of the arc *AB*, is perpendicular to the radius *AO* (110); and the tangent *AG*, drawn in the plane of the arc *AC*, is perpendicular to the same radius *AO*; therefore the angle *FAG* is equal to the angle of the planes *OAB*, *OAC* (349), which is that of the arcs *AB*, *AC*, and which is designated by *BAC*.

In like manner, if the arc *AD* is equal to a quadrant, and also *AE*, the lines *OD*, *OE*, will be perpendicular to *AO*, and the angle *DOE* will be equal to the angle of the planes *AOD*, *AOE*; therefore the arc *DE* is the measure of the angle of these planes, or the measure of the angle *CAB*.

472. *Corollary*. The angles of spherical triangles may be compared with each other by means of the arcs of great circles, described from their vertices as poles and comprehended between their sides; thus it is easy to make an angle equal to a given angle.

473. *Scholium*. The angles opposite to each other at the Fig. 238. vertex, as *ACO*, *BCN* (*fig*. 238), are equal; for each is equal to the angle formed by the two planes *ACB*, *OCN* (350).

It will be perceived also that in the meeting of two arcs *ACB*, *OCN*, the two adjacent angles *ACO*, *OCB*, taken together, are equal to two right angles.

THEOREM.

Fig. 227. 474. *The triangle* ABC (fig. 227) *being given, if, from the points* A, B, C, *as poles, the arcs* EF, FD, DE, *be described, forming the triangle* DEF, *reciprocally the points* D, E, F, *will be the poles of the sides* BC, AC, AB.

Demonstration. The point *A* being the pole of the arc *EF*, the distance *AE* is a quadrant; the point *C* being the pole of the arc *DE*, the distance *CE* is likewise a quadrant; consequently the point *E* is distant a quadrant from each of the points *A*, *C*; therefore it is the pole of the arc *AC* (467). It may be shown, in the same manner, that *D* is the pole of the arc *BC*, and *F* that of the arc *AB*.

475. *Corollary*. Hence the triangle *ABC* may be described by means of *DEF*, as *DEF* is described by means of *ABC*.

THEOREM.

476. *The same things being supposed as in the preceding theorem,* Fig. 227. *each angle of one of the triangles* ABC, DEF (fig. 227), *will have for its measure a semicircumference minus the side opposite in the other triangle.*

Demonstration. Let the sides AB, AC, be produced, if necessary, till they meet EF in G and H; since the point A is the pole of the arc GH, the angle A will have for its measure the arc GH. But the arc EH is a quadrant, as also GF, since E is the pole of AH, and F the pole of AG (465); consequently $EH + GF$ is equal to a semicircumference. But $EH + GF$ is the same as $EF + GH$; therefore the arc GH, which measures the angle A, is equal to a semicircumference minus the side EF; likewise the angle B has for its measure $\frac{1}{2}$ *circ.* — DF, and the angle C, $\frac{1}{2}$ *circ.* — DE.

This property must be reciprocal between the two triangles, since they are described in the same manner, the one by means of the other. Thus we shall find that the angles D, E, F, of the triangle DEF have for their measure respectively $\frac{1}{2}$ *circ.* — BC, $\frac{1}{2}$ *circ.* — AC, $\frac{1}{2}$ *circ.* — AB. Indeed, the angle D, for example, has for its measure the arc MI; but

$$MI + BC = MC + BI = \tfrac{1}{2}\,\textit{circ.};$$

therefore the arc MI, the measure of the angle D, $= \frac{1}{2}$ *circ.* — BC, and so of the others.

477. *Scholium.* It may be remarked that, beside the triangle DEF, three others may be formed by the intersection of the three arcs DE, EF, DF. But the proposition applies only to the central triangle, which is distinguished from the three others by this, that the two angles A, D, are situated on the same side of BC, the two B, E, on the same side of AC, and the two C, F, on the same side of AB.

Different names are given to the triangles ABC, DEF; we shall call them *polar triangles.*

LEMMA.

478. *The triangle* ABC (fig. 229) *being given, if, from the pole* A *and with the distance* AC, *an arc of a small circle* DEC *be described, if, also from the pole* B *and with the distance* BC, *the arc* DFC *be described, and from the point* D *where the arcs* DEC, DFC, *cut each other, the arcs of great circles* AD, DB, *be drawn; we say that of the triangle* ADB *thus formed the parts will be equal to those of the triangle* ACB. Fig. 229

Demonstration. For, by construction the side $AD = AC$, $DB = BC$, and AB is common; therefore the two triangles will have the sides equal, each to each. We say, moreover, that the angles opposite to the equal sides are equal.

Indeed, if the centre of the sphere be supposed in *O*, we can suppose a solid angle formed at the point *O* by the three plane angles *AOB*, *AOC*, *BOC*; we can suppose likewise a second solid angle formed by the three plane angles *AOB*, *AOD*, *BOD*. And since the sides of the triangle *ABC* are equal to those of the triangle *ADB*, it follows that the plane angles, which form one of the solid angles, are equal to the plane angles which form the other solid angle, each to each. But the planes of any two angles in the one solid have the same inclination to each other as the planes of the homologous angles in the other (359); consequently the angles of the spherical triangle *DAB* are equal to those of the triangle *CAB*, namely, $DAB = BAC$, $DBA = CBA$, and $ADB = ACB$; therefore the sides and the angles of the triangle *ADB* are equal to the sides and angles of the triangle *ACB*.

479. *Scholium.* The equality of these triangles does not depend upon an absolute equality, or equality by superposition, for it would be impossible to make them coincide by applying the one to the other, at least except they should happen to be isosceles. The equality then under consideration is that which we have already called equality by *symmetry*, and for this reason, we shall call the triangles *ACB*, *ADB*, *symmetrical triangles*.

THEOREM.

480. *Two triangles situated on the same sphere, or on equal spheres, are equal in all their parts, when two sides and the included angle of the one are equal to two sides and the included angle of the other, each to each.*

Fig. 230. *Demonstration.* Let the side $AB = EF$ (*fig.* 230), the side $AC = EG$, and the angle $BAC = FEG$, the triangle *EFG* can be placed upon the triangle *ABC*, or upon the triangle symmetrical with it *ABD*, in the same manner as two plane triangles are applied, when they have two sides and the included angle of the one respectively equal to two sides and the included angle of the other (36). Therefore all the parts of the triangle *EFG* will be equal to those of the triangle *ABC*, that is, beside the three parts which were supposed equal we shall have the side $BC = FG$, the angle $ABC = EFG$, and the angle $ACB = EGF$.

THEOREM.

481. *Two triangles situated on the same sphere, or on equal spheres, are equal in all their parts, when a side and the two adjacent angles of the one are equal to a side and the two adjacent angles of the other, each to each.*

Demonstration. For one of these triangles may be applied to the other as has been done in the analogous case of plane triangles (38).

THEOREM.

482. *If two triangles situated on the same sphere, or on equal spheres, are equilateral with respect to each other, they will also be equiangular with respect to each other, and the equal angles will be opposite to equal sides.*

Demonstration. This proposition is manifest from the reasoning pursued in art. 478, by which it is shown that with three given sides *AB*, *AC*, *BC*, only two triangles can be constructed, differing as to the position of their parts, but equal as to the magnitude of these parts. Therefore two triangles, which are equilateral with respect to each other, are either absolutely equal, or at least equal by symmetry; in either case they are equiangular with respect to each other and the equal angles are opposite to equal sides.

THEOREM.

483. *In every isosceles spherical triangle the angles opposite to the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.*

Demonstration. 1. Let *AB* be equal to *AC* (*fig.* 231); we say that the angle *C* will be equal to the angle *B*. For, if from the vertex *A* the arc *AD* be drawn to the middle of the base, the two triangles *ABD*, *ADC*, will have the three sides of the one equal to the three sides of the other, each to each, namely, *AD* common, $BD = DC$, $AB = AC$; consequently, by the preceding theorem, the two triangles will have their homologous angles equal, therefore $B = C$. Fig. 231.

2. Let the angle *B* be equal to *C*; we say that *AB* will be equal to *AC*. For, if the side *AB* is not equal to *AC*, let *AB* be

the greater; take $BO = AC$, and join OC. The two sides BO, BC, are equal to the two AC, BC; and the angle OBC contained by the first is equal to the angle ACB contained by the second. Consequently the two triangles have their other parts equal (480), namely $OCB = ABC$; but the angle ABC is, by hypothesis, equal to ACB; whence OCB is equal to ACB, which is impossible; AB then cannot be supposed unequal to AC; therefore the sides AB, AC, opposite to the equal angles B, C, are equal.

484. *Scholium.* It is evident, from the same demonstration, that the angle $BAD = DAC$, and the angle $BDA = ADC$. Consequently the two last are right angles; therefore, *the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, is perpendicular to this base, and divides the angle opposite into two equal parts.*

THEOREM.

Fig. 232. 485. *In any spherical triangle* ABC (fig. 232), *if the angle* A *is greater than the angle* B, *the side* BC *opposite to the angle* A *will be greater than the side* AC *opposite to the angle* B; *conversely, if the side* BC *is greater than* AC, *the angle* A *will be greater than the angle* B.

Demonstration. 1. Let the angle $A > B$; make the angle $BAD = B$, and we shall have $AD = DB$ (483); but

$$AD + DC > AC;$$

in the place of AD substitute DB, and we shall have $DB + DC$ or $BC > AC$.

2. If we suppose $BC > AC$, we say that the angle BAC will be greater than ABC. For, if BAC were equal to ABC, we should have $BC = AC$; and if BAC were less than ABC, it would follow, according to what has just been demonstrated, that $BC < AC$, which is contrary to the supposition, therefore the angle BAC is greater than ABC.

THEOREM.

Fig. 233. 486. *If the two sides* AB, AC (fig. 233), *of the spherical triangle* ABC *are equal to the two sides* DE, DF, *of the triangle* DEF *described upon an equal sphere, if at the same time the angle* A *is greater than the angle* D, *we say that the third side* BC *of the first triangle will be greater than the third side* EF *of the second.*

The demonstration of this proposition is entirely similar to that of art. 42.

THEOREM.

487. *If two triangles described upon the same sphere or upon equal spheres are equiangular with respect to each other, they will also be equilateral with respect to each other.*

Demonstration. Let A, B, be the two given triangles, P, Q, their polar triangles. Since the angles are equal in the triangles A, B, the sides will be equal in the polar triangles P, Q, (476); but, since the triangles P, Q, are equilateral with respect to each other, they are also equiangular with respect to each other (482); and, the angles being equal in the triangles P, Q, it follows that the sides are equal in their polar triangles A, B. Therefore the triangles A, B, which are equiangular with respect to each other, are at the same time equilateral with respect to each other.

This proposition may be demonstrated without making use of polar triangles in the following manner.

Let ABC, DEF (*fig.* 234), be two triangles equiangular with respect to each other, having $A = D$, $B = E$, $C = F$; we say that the sides will be equal, namely, $AB = DE$, $AC = DF$, $BC = EF$. Fig. 234.

Produce AB. AC, making $AG = DE$, $AH = DF$; join GH, and produce the arcs BC, GH, till they meet in I and K.

The two sides AG. AH, are, by construction, equal to the two DF, DE, the included angle $GAH = BAC = EDF$, consequently the triangles AGH, DEF, are equal in all their parts (480); therefore the angle $AGH = DEF = ABC$, and the angle

$$AHG = DFE = ACB.$$

In the triangles IBG, KBG, the side BG is common, and the angle $IGB = GBK$; and, since $IGB + BGK$ is equal to two right angles, as also $GBK + IBG$, it follows that $BGK = IBG$. Consequently the triangles IBG, GBK, are equal (481); therefore $IG = BK$, and $IB = GK$.

In like manner, since the angle $AHG = ACB$, the triangles ICH, HCK, have a side and the two adjacent angles of the one respectively equal to a side and the two adjacent angles of the other; consequently they are equal; therefore $IH = CK$, and $HK = IC$.

Now, if from the equals *BK*, *IG*, we take the equal *CK*, *IH*, the remainders *BC*, *GH*, will be equal. Besides, the angle $BCA = AHG$, and the angle $ABC = AGH$. Whence the triangles *ABC*, *AHG*, have a side and the two adjacent angles of the one respectively equal to a side and the two adjacent angles of the other; consequently they are equal. But the triangle *DEF* is equal in all its parts to the triangle *AHG*; therefore it is also equal to the triangle *ABC*, and we shall have $AB = DE$, $AC = DF$, $BC = EF$; hence, if two spherical triangles are equiangular with respect to each other, the sides opposite to the equal angles will be equal.

488. *Scholium.* This proposition does not hold true with regard to plane triangles, in which from the equality of the angle we can only infer the proportionality of the sides. But it is easy to account for the difference in this respect between plane and spherical triangles. In the present proposition, as well as those of articles 480, 481, 482, 486, which relate to a comparison of triangles, it is said expressly that the triangles are described upon the same sphere or upon equal spheres. Now similar arcs are proportional to their radii; consequently upon equal spheres two triangles cannot be similar without being equal. It is not therefore surprising that equality of angles should imply equality of sides.

It would be otherwise, if the triangles were described upon unequal spheres; then, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of the spheres.

THEOREM.

489. *The sum of the angles of every spherical triangle is less than six and greater than two right angles.*

Demonstration. 1. Each angle of a spherical triangle is less than two right angles (*see the following scholium*); therefore the sum of the three angles is less than six right angles.

2. The measure of each angle of a spherical triangle is equal to the semicircumference minus the corresponding side of the polar triangle (476); therefore the sum of the three angles has for its measure three semicircumferences minus the sum of the sides of the polar triangle. Now this last sum is less than a

circumference (461); consequently, by subtracting it from three semicircumferences the remainder will be greater than a semicircumference, which is the measure of two right angles; therefore the sum of the three angles of a spherical triangle is greater than two right angles.

490. *Corollary* I. The sum of the angles of a spherical triangle is not constant like that of a plane triangle; it varies from two right angles to six, without the possibility of being equal to either limit. Thus, two angles being given, we cannot thence determine the third.

491. *Corollary* II. A spherical triangle may have two or three right angles, also two or three obtuse angles.

If the triangle *ABC* (*fig.* 235) has two right angles *B* and *C*, the vertex *A* will be the pole of the base *BC* (467); and the sides *AB*, *AC*, will be quadrants. Fig. 235.

If the angle *A* also is a right angle, the triangle *ABC* will have all its angles right angles, and all its sides quadrants. The triangle having three right angles is contained eight times in the surface of the sphere; this is evident from figure 236, if we suppose the arc *MN* equal to a quadrant.

492. *Scholium.* We have supposed in all that precedes, conformably to the definition art. 442, that spherical triangles always have their sides less each than a semicircumference; then it follows that the angles are always less than two right angles. For the side *AB* (*fig.* 224) is less than a semicircumference as also *AC*; these arcs must both be produced in order to meet in *D*. Now the two angles *ABC*, *CBD*, taken together, are equal to two right angles; therefore the angle *ABC* is by itself less than two right angles. Fig. 224.

We will remark, however, that there are spherical triangles of which certain sides are greater than a semicircumference, and certain angles greater than two right angles. For, if we produce the side *AC* till it becomes an entire circumference *ACE*, what remains, after taking from the surface of the hemisphere the triangle *ABC*, is a new triangle, which may also be designated by *ABC*, and the sides of which are *AB*, *BC*, *AEDC*. We see then, that the side *AEDC* is greater than the semicircumference *AED*; but, at the same time, the opposite angle *B* exceeds two right angles by the quantity *CBD*.

Besides, if we exclude from the definition triangles, the sides and angles of which are so great, it is because the resolution of them, or the determination of their parts, reduces itself always to that of triangles contained in the definition. Indeed, it will be readily perceived, that if we know the angles and sides of the triangle *ABC*, we shall know immediately the angles and sides of the triangle of the same name, which is the remainder of the surface of the hemisphere.

THEOREM.

Fig. 236. 493. *The lunary surface* AMBNA (fig. 236) *is to the surface of the sphere as the angle* MAN *of this surface is to four right angles, or as the arc* MN, *which measures this angle, is to the circumference.*

Demonstration. Let us suppose in the first place, that the arc *MN* is to the circumference *MNPQ* in the ratio of two entire numbers, as 5 to 48, for example. The circumference *MNPQ* may be divided into 48 equal parts, of which *MN* will contain 5; then joining the pole *A* and the points of division by as many quadrants, we shall have 48 triangles in the surface of the hemisphere *AMNPQ*, which will be equal among themselves, since they have all their parts equal. The entire sphere then will contain 96 of these partial triangles, and the lunary surface *AMBNA* will contain 10 of them; therefore the lunary surface is to that of the sphere as 10 is to 96, or as 5 is to 48, that is, as the arc *MN* is to the circumference.

If the arc *MN* is not commensurable with the circumference, it may be shown by a course of reasoning, of which we have already had many examples, that the lunary surface is always to that of the sphere as the arc *MN* is to the circumference.

494. *Corollary* I. Two lunary surfaces are to each other as their respective angles.

495. *Corollary* II. We have already seen that the entire surface of the sphere is equal to eight triangles having each three right angles (491); consequently, if the area of one of these triangles be taken for unity, the surface of the sphere will be represented by eight. This being supposed, the lunary surface, of which the angle is *A*, will be expressed by $2A$, the angle *A* being estimated by taking the right angle for unity; for we have $2A : 8 :: A : 4$. Here are then two kinds of units; one for

angles, this is the right angle; the other for surfaces, this is the spherical triangle, of which all the angles are right angles and the sides quadrants.

496. *Scholium.* The spherical wedge comprehended by the planes AMB, ANB, is to the entire sphere, as the angle A is to four right angles. For the lunary surfaces being equal, the spherical wedges will also be equal; therefore two spherical wedges are to each other as the angles formed by the planes which comprehend them.

THEOREM.

497. *Two symmetrical spherical triangles are equal in surface.*

Demonstration. Let ABC, DEF (*fig.* 237), be two symmetrical triangles, that is, two triangles which have their sides equal, namely, $AB = DE$, $AC = DF$, $CB = EF$, and which at the same time do not admit of being applied the one to the other; we say that the surface ABC is equal to the surface DEF. Fig. 237.

Let P be the pole of the small circle which passes through the three points A, B, C*; from this point draw the equal arcs PA, PB, PC (464); at the point F make the angle $DFQ = ACP$, the arc $FQ = CP$, and join DQ, EQ.

The sides DF, FQ, are equal to the sides AC, CP; the angle $DFQ = ACP$; consequently the two triangles DFQ, ACP, are equal in all their parts (480); therefore the side $DQ = AP$, and the angle $DQF = APC$.

In the proposed triangles DFE, ABC, the angles DFE, ACB, opposite to the equal sides DE, AB, being equal (481), if we subtract from them the angles DFQ, ACP, equal, by construction, there will remain the angle QFE equal to PCB. Moreover the sides QF, FE, are equal to the sides PC, CB; consequently the two triangles FQE, CPB, are equal in all their parts; therefore the side $QE = PB$, and the angle $FQE = CPB$.

If we observe now that the triangles DFQ, ACP, which have the sides equal each to each, are at the same time isosceles, we shall perceive that they may be applied the one to the other; for, having placed *PA* upon its equal *QF*, the side *PC* will fall upon its equal *QD*, and thus the two triangles will coincide; consequently they are equal, and the surface $DQF = APC$. For a similar reason the surface $FQE = CPB$, and the surface $DQE = APB$; we have accordingly

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* The circle, which passes through the three points A, B, C, or which is circumscribed about the triangle ABC, can only be a small circle of the sphere; for, if it were a great one, the three sides AB, BC, AC, would be situated in the same plane, and the triangle ABC would be reduced to one of its sides.

$DQF + FQE - DQE = APC + CPB - APB$, or $DEF = ABC$; therefore the two symmetrical triangles *ABC*, *DEF*, are equal in surface.

498. *Scholium.* The poles *P* and *Q* may be situated within the triangles *ABC*, *DEF*; then it would be necessary to add the three triangles *DQF*, *FQE*, *DQE*, in order to obtain the triangle *DEF*, and also the three triangles *APC*, *CPB*, *APB*, in order to obtain the triangle *ABC*. In other respects the demonstration would always be the same and the conclusion the same.

THEOREM.

Fig. 238. 499. *If two great circles* AOB, COD (fig. 238), *cut each other in any manner in the surface of a hemisphere* AOCBD, *the sum of the opposite triangles* AOC, BOD, *will be equal to the lunary surface of which the angle is* BOD.

Demonstration. By producing the arcs *OB*, *OD*, into the surface of the other hemisphere till they meet in *N*, *OBN* will be a semicircumference as well as *AOB*; taking from each *OB*, we shall have $BN = AO$. For a similar reason $DN = CO$, and $BD = AC$; consequently the two triangles *AOC*, *BDN*, have the three sides of the one equal respectively to the three sides of the other; moreover, their position is such that they are symmetrical; therefore they are equal in surface (496), and the sum of the triangles *AOC*, *BOD*, is equivalent to the lunary surface *OBNDO*, of which the angle is *BOD*.

500. *Scholium.* It is evident also that the two spherical pyramids, which have for their bases the triangles *AOC*, *BOD*, taken together, are equal to the spherical wedge of which the angle is *BOD*.

THEOREM.

501. *The surface of a spherical triangle has for its measure the excess of the sum of the three angles over two right angles.*

Demonstration. Let ABC (*fig.* 239) be the triangle proposed; produce the sides till they meet the great circle $DEFG$ drawn at pleasure without the triangle. By the preceding theorem the two triangles ADE, AGH, taken together, are equal to the lunary surface of which the angle is A, and which has for its measure $2A$ (495); thus we shall have $ADE + AGH = 2A$; for a similar reason $BGF + BID = 2B$, $CIH + CFE = 2C$. But the sum of these six triangles exceeds the surface of a hemisphere by twice the triangle ABC; moreover the surface of a hemisphere is represented by 4; consequently the double of the triangle ABC is equal to $2A + 2B + 2C - 4$, and consequently $ABC = A + B + C - 2$; therefore every spherical triangle has for its measure the sum of its angles minus two right angles. Fig. 239.

502. *Corollary* I. The proposed triangle will contain as many triangles of three right angles, or eighths of the sphere (494), as there are right angles in the measure of this triangle. If the angles, for example, are each equal to $\frac{4}{3}$ of a right angle, then the three angles will be equal to four right angles, and the proposed triangle will be represented by $4 - 2$ or 2; therefore it will be equal to two triangles of three right angles, or to a fourth of the surface of the sphere.

503. *Corollary* II. The spherical triangle ABC is equivalent to a lunary surface, the angle of which is $\frac{A+B+C}{2} - 1$; likewise the spherical pyramid, the base of which is ABC, is equal to the spherical wedge, the angle of which is $\frac{A+B+C}{2} - 1$.

504. *Scholium.* At the same time that we compare the spherical triangle ABC with the triangle of three right angles, the spherical pyramid, which has for its base ABC, is compared with the pyramid which has a triangle of three right angles for its base, and we obtain the same proportion in each case. The solid angle at the vertex of a pyramid is compared in like manner with the solid angle at the vertex of the pyramid having a triangle of three right angles for its base. Indeed the comparison is established by the coincidence of the parts. Now, if the bases of pyramids coincide, it is evident that the pyramids themselves will coincide, as also the solid angles at the vertex. Whence we derive several consequences;

1. Two spherical triangular pyramids are to each other as their bases; and, since a polygonal pyramid may be divided into several triangular pyramids, it follows that any two spherical pyramids are to each other as the polygons which constitute their bases.

2. The solid angles at the vertex of these same pyramids are likewise proportional to the bases; therefore, in order to compare any two solid angles, the vertices are to be placed at the centres of two equal spheres, and these solid angles will be to each other as the spherical polygons intercepted between their planes or faces.

The angle at the vertex of the pyramid, whose base is a triangle of three right angles, is formed by three planes perpendicular to each other; this angle, which may be called a *solid right angle*, is very proper to be used as the unit of measure for other solid angles. This being supposed, the same number, which gives the area of a spherical polygon, will give the measure of the corresponding solid angle. If, for example, the area of a spherical polygon is $\frac{3}{4}$, that is, if it is $\frac{3}{4}$ of a triangle of three right angles, the corresponding solid angle will also be $\frac{3}{4}$ of a solid right angle.

THEOREM.

505. *The surface of a spherical polygon has for its measure the sum of its angles minus the product of two right angles by the number of sides in the polygon minus two.*

Fig. 240. *Demonstration.* From the same vertex A (*fig.* 240) let there be drawn to the other vertices the diagonals AC, AD; the polygon $ABCDE$ will be divided into as many triangles minus two as it has sides. But the surface of each triangle has for its measure the sum of its angles minus two right angles, and it is evident that the sum of all the angles of the triangles is equal to the sum of the angles of the polygon; therefore the surface of the polygon is equal to the sum of its angles diminished by as many times two right angles as there are sides minus two.

506. *Scholium.* Let s be the sum of the angles of a spherical polygon, n the number of its sides; the right angle being supposed unity, the surface of the polygon will have for its measure $s - 2n \quad - 2)$ or $s - 2n + 4$.

SECTION FOUTH.

Of the Three Round Bodies.

DEFINITIONS.

507. We call a *cylinder* the solid generated by the revolution of a rectangle *ABCD* (*fig.* 250), which may be conceived to turn about the side *AB* considered as fixed. Fig. 250.

During this revolution the sides *AD*, *BC*, remaining always perpendicular to *AB*, describe equal circular planes *DHP*, *CGQ*, which are called the *bases* of the cylinder, and the side *CD* describes the *convex surface of the cylinder*.

The fixed line *AB* is called the *axis of the cylinder*.

Every section *KLM*, made by a plane perpendicular to the axis, is a circle equal to each of the bases; for, while the rectangle *ABCD* turns about *AB*, the line *IK*, perpendicular to *AB*, describes a circular plane equal to the base, and this plane is simply the section made perpendicular to the axis at the point *I*.

Every section *PQGH*, made by a plane passing through the axis, is a rectangle double of the generating rectangle *ABCD*.

508. We call a *cone* the solid generated by the revolution of a right-angled triangle *SAB* (*fig.* 251), which may be conceived to turn about the fixed side *SA*. Fig. 251.

In this revolution the side *AB* describes a circular plane *BDCE* called the *base of the cone*, and the hypothenuse *SB* describes the *convex surface of the cone*.

The point *S* is called the *vertex of the cone*, *SA* the *axis* or *altitude*, and *SB* the *side*.

Every section *HKFI*, made perpendicularly to the axis, is a circle; every section *SDE*, made through the axis, is an isosceles triangle double of the generating triangle *SAB*.

509. If from the cone *SCDB* we separate by a section parallel to the base the cone *SFKH*, the remaining solid *CBHF* is called a *truncated cone* or a *frustum of a cone*. It may be conceived to be generated by the revolution of the trapezoid *ABHG*, of which the angles *A* and *G* are right angles, about the side *AG*. The fixed line *AG* is called the axis or altitude of the frustum, the circles *BDC*, *HKF*, are the *bases* and *BH* the *side* of the frustum.

510. Two cylinders or two cones are *similar*, when their axes are to each other as the diameters of their bases.

Fig. 252. 511. If, in the circle *ACD* (*fig.* 252), considered as the base of a cylinder, a polygon *ABCDE* be inscribed, and upon the base *ABCDE* a right prism be erected equal in altitude to the cylinder, the prism is said to be *inscribed in the cylinder*, or the cylinder to be *circumscribed about the prism.*

It is manifest that the edges *AF*, *BG*, *CH*, &c., of the prism, being perpendicular to the plane of the base, are comprehended in the convex surface of the cylinder; therefore the prism and cylinder touch each other along these lines.

Fig. 253. 512. In like manner, if *ABCD* (*fig.* 253) be a polygon circumscribed about the base of a cylinder, and upon the base *ABCD* a right prism, equal in altitude to the cylinder, be constructed, the prism is said to be *circumscribed about the cylinder*, or the cylinder *inscribed in the prism.*

Let *M*, *N*, &c., be the points of contact of the sides *AB*, *BC*, &c., and through the points *M*, *N*, &c., let the lines *MX*, *NY*, &c., be drawn perpendicular to the plane of the base; it is evident that these perpendiculars will be in the surface of the cylinder and in that of the circumscribed prism at the same time; therefore they will be lines of contact.

N. B. The cylinder, the cone and the sphere are the *three round bodies*, which are treated of in the elements.

Preliminary Lemmas upon Surfaces.

Fig. 254. 513. I. *A plane surface* OABCD (fig. 254) *is less than any other surface* PABCD *terminated by the same perimeter ABCD.*

Demonstration. This proposition is sufficiently evident to be ranked among the number of axioms; for we may consider the plane among surfaces what the straight line is among lines. The straight line is the shortest distance between two given points; in like manner the plane is the least surface among all those which have the same perimeter. Still, as it is proper to make the number of axioms as small as possible, I shall present a process of reasoning which will leave no doubt with regard to this proposition.

As a surface is extension in length and breadth, we cannot conceive one surface to be greater than another, except the dimensions of the first exceed in some direction those of the second; and if it happens that the dimensions of one surface are in all directions less than the dimensions of another surface, it is evident that the first surface will be less than the second. Now, in whatever direction the plane *BPD* be made to pass, as it cuts the plane in *BD*, and the other surface in *BPD*, the straight line *BD*, will always be less than *BPD*; therefore the plane surface *OABCD* is less than the surrounding surface *PABCD*.

514. II. *A* convex *surface* OABCD (*fig.* 255) *is less than any other surface which encloses it by resting on the same perimeter* ABCD. Fig. 255.

Demonstration. We repeat here, that we understand by a *convex surface* a surface that cannot be met by a straight line in more than two points; still it is possible that a straight line may apply itself exactly to a convex surface in a certain direction; we have examples of this in the surfaces of the cone and cylinder. It should be observed moreover, that the denomination of *convex surface* is not confined to curved surfaces; it comprehends *polyedral* faces, or surfaces composed of several planes, also surfaces that are in part curved and in part polyedral.

This being premised, if the surface *OABCD* is not smaller than any of those which enclose it, let there be among these last *PABCD* the smallest surface which shall be at most equal to *OABCD*. Through any point *O* suppose a plane to pass touching the surface *OABCD* without cutting it; this plane will meet the surface *PABCD*, and the part which it separates from it will be greater than the plane terminated by the same surface; therefore by preserving the rest of the surface *PABCD*, we can substitute the plane for the part taken away, and we shall have a new surface, which encloses the surface *OABCD*, and which would be less than *PABCD*. But this last is the least of all, by hypothesis; consequently this hypothesis cannot be maintained; therefore the convex surface *OABCD* is less than any which encloses *OABCD* and which is terminated by the same perimeter *ABCD*.

515. *Scholium.* By a course of reasoning entirely similar it may be shown,

1. That if a convex surface terminated by two perimeters, *ABC*, *DEF* (*fig.* 256), is enclosed by any other surface terminated by the same perimeters, the enclosed surface will be less than the other. Fig. 256.

Fig. 257. 2. That, if a convex surface *AB* (*fig.* 257) is enclosed on all sides by another surface *MN*, whether they have points, lines, or planes in common, or whether they have no point in common, the enclosed surface is always less than the enclosing surface.

For among these last there cannot be one which shall be the least of all, since in all cases we can draw the plane *CD* a tangent to the convex surface, which plane would be less than the surface *CMD*; and thus the surface *CND* would be smaller than *MN*, which is contrary to the hypothesis, that *MN* is the smallest of all. Therefore the convex surface *AB* is less than any which encloses it.

THEOREM.

516. *The solidity of a cylinder is equal to the product of its base by its altitude.*

Fig. 258. *Demonstration.* Let *CA* (*fig.* 258) be the radius of the base of the given cylinder, *H* its altitude; and let *surf. CA* represent the surface of a circle whose radius is *CA*; we say that the solidity of the cylinder will be *surf.* $CA \times H$. For, if *surf.* $CA \times H$ is not the measure of the given cylinder, this product will be the measure of a cylinder either greater or less. In the first place let us suppose that it is the measure of a less cylinder, of a cylinder, for example, of which *CD* is the radius of the base and *H* the altitude.

Circumscribe about the circle, of which *CD* is the radius, a regular polygon *GHIP*, the sides of which shall not meet the circumference of which *CA* is the radius (285); then suppose a right prism having for its base the polygon *GHIP*, and for its altitude *H*, this prism will be circumscribed about the cylinder of which the radius of the base is *CD*. This being premised, the solidity of the prism is equal to the product of its base *GHIP* multiplied by the altitude *H*; and the base *GHIP* is less than the circle whose radius is *CA*; therefore the solidity of the prism is less than *surf.* $CA \times H$. But *surf.* $CA \times H$ is, by hypothesis, the solidity of the cylinder inscribed in the prism; consequently the prism would be less than the cylinder; but the cylinder on the contrary is less than the prism, because it is contained in it; therefore it is impossible that *surf.* $CA \times H$ should be the measure of a cylinder of which the radius of the base is *CD* and the altitude H; or in more general terms *the product of the base of a cylinder by its altitude cannot be the measure of a less cylinder.*

We say, in the second place, that this same product cannot be the measure of a greater cylinder; for, not to multiply figures, let CD be the radius of the base of the given cylinder; and, if it be possible, let *surf.* $CD \times H$ be the measure of a greater cylinder, of a cylinder, for example, of which CA is the radius of the base and H the altitude.

The same construction being supposed as in the first case, the prism circumscribed about the given cylinder will have for its measure $GHIP \times H$; the area $GHIP$ is greater than *surf.* CD; consequently, the solidity of the prism in question is greater than *surf.* $CD \times H$; the prism then would be greater than the cylinder of the same altitude whose base is *surf.* CA. But the prism on the contrary is less than the cylinder, since it is contained in it; therefore *it is impossible that the product of the base of a cylinder by its altitude should be the measure of a greater cylinder.*

We conclude then, that the solidity of a cylinder is equal to the product of its base by its altitude.

517. *Corollary* I. Cylinders of the same altitude are to each other as their bases, and cylinders of the same base are to each other as their altitudes.

518. *Corollary* II. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of the bases. For the bases are as the squares of their diameters; and, since the cylinders are similar, the diameters of the bases are as the altitudes (510); consequently the bases are as the squares of the altitudes; therefore the bases multiplied by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

519. *Scholium.* Let R be the radius of the base of a cylinder, H its altitude, the surface of the base will be πR^2 (291), and the solidity of the cylinder will be $\pi R^2 \times H$, or $\pi R^2 H$.

LEMMA.

520. *The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.*

Demonstration. This surface is equal to the sum of the rectangles $AFGB$, $BGHC$, $CHID$, &c. (*fig.* 252), which compose it. Fig. 252.

Now the altitudes *AF*, *BG*, *CH*, &c., of these rectangles are each equal to the altitude of the prism. Therefore the sum of these rectangles, or the convex surface of the prism, is equal to the perimeter of its base multiplied by its altitude.

521. *Corollary.* If two right prisms have the same altitude, the convex surfaces of these prisms will be to each other as the perimeters of the bases.

LEMMA.

522. *The convex surface of a cylinder is greater than the convex surface of any inscribed prism, and less than the convex surface of any circumscribed prism.*

Demonstration. The convex surface of the cylinder and that of Fig 252. the inscribed prism *ABCDEF* (*fig.* 252) may be considered as having the same length, since every section made in the one and the other parallel to *AF* is equal to *AF*; and if, in order to obtain the magnitude of these surfaces, we suppose them to be cut by planes parallel to the base, or perpendicular to the edge *AF*, the sections will be equal, the one to the circumference of the base, and the other to the perimeter of the polygon *ABCDE* less than this circumference; since therefore, the lengths being equal, the breadth of the cylindric surface is greater than that of the prismatic surface, it follows that the first surface is greater than the second.

By a course of reasoning entirely similar it may be shown that the convex surface of the cylinder is less than that of any Fig. 253. circumscribed prism *BCDKLH* (*fig.* 253).

THEOREM.

523. *The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude.*

Fig. 258. *Demonstration.* Let *CA* (*fig.* 258) be the radius of the base of the given cylinder, *H* its altitude; and let *circ.* *CA* be the circumference of a circle whose radius is *CA*; we say that *circ.* $CA \times H$ will be the convex surface of the cylinder. For, if this proposition be denied, *circ.* $CA \times H$ must be the surface of a cylinder either greater or less; and, in the first place, let us suppose that it is the surface of a less cylinder, of a cylinder, for example, of which the radius of the base is *CD* and the altitude *H*.

Circumscribe about the circle, whose radius is *CD*, a regular polygon *GHIP*, the sides of which shall not meet the circumference whose radius is *CA*; then suppose a right prism, whose altitude is *H*, and whose base is the polygon *GHIP*. The convex surface of this prism will be equal to the perimeter of the polygon *GHIP* multiplied by its altitude *H* (520); this perimeter is less than the circumference of the circle whose radius is *CA*; consequently the convex surface of the prism is less than *circ.* $CA \times H$. But *circ.* $CA \times H$ is, by hypothesis, the convex surface of a cylinder of which *CD* is the radius of the base, which cylinder is inscribed in the prism; whence the convex surface of the prism would be less than that of the inscribed cylinder. But on the contrary it is greater (522); accordingly the hypothesis with which we set out is absurd; therefore, 1. *the circumference of the base of a cylinder multiplied by its altitude cannot be the measure of the convex surface of a less cylinder.*

We say, in the second place, that this same product cannot be the measure of the surface of a greater cylinder. For, not to change the figure, let *CD* be the radius of the base of the given cylinder, and, if it be possible, let *circ.* $CD \times H$ be the convex surface of a cylinder, which with the same altitude has for its base a greater circle, the circle, for example, whose radius is *CA*. The same construction being supposed as in the first hypothesis, the convex surface of the prism will always be equal to the perimeter of the polygon *GHIP*, multiplied by the altitude *H*. But this perimeter is greater than *circ.* *CD*; consequently the surface of the prism would be greater than *circ.* $CD \times H$, which, by hypothesis, is the surface of a cylinder of the same altitude of which *CA* is the radius of the base. Whence the surface of the prism would be greater than that of the cylinder. But while the prism is inscribed in the cylinder, its surface will be less than that of the cylinder (522); for a still stronger reason is it less when the prism does not extend to the cylinder; consequently the second hypothesis cannot be maintained; therefore, 2. *the circumference of the base of a cylinder multiplied by its altitude cannot be the measure of the surface of a greater cylinder.*

We conclude then that the convex surface of a cylinder is equal to the circumference of the base multiplied by its altitude.

$\pi \times \overline{AO}^2$; likewise the base $IKL = \pi \times \overline{DP}^2$, and the mean proportional between $\pi \times \overline{AO}^2$ and $\pi \times \overline{DP}^2$ is $\pi AO \times DP$; therefore the solidity of the frustum of a pyramid or that of the frustum of a cone has for its measure

$$\tfrac{1}{3} OP \times (\pi \times \overline{AO}^2 + \pi \times \overline{DP}^2 + \pi \times AO \times DP) \quad (422),$$

or

$$\tfrac{1}{3} \pi \times OP \times (\overline{AO}^2 + \overline{PD}^2 + AO \times DP).$$

THEOREM.

528. *The convex surface of a cone is equal to the circumference of its base multiplied by half its side.*

Fig. 259. *Demonstration.* Let AO (*fig.* 259), be the radius of the base of the given cone, S its vertex, and SA its side; we say that the surface will be *circ.* $AO \times \tfrac{1}{2} SA$. For, if it be possible, let *circ.* $AO \times \tfrac{1}{2} SA$ be the surface of a cone which has S for its vertex, and for its base the circle described with a radius OB greater than AO.

Circumscribe about the small circle a regular polygon $MNPT$, the sides of which shall not meet the circumference of which OB is the radius; and let $SMNPT$ be a regular pyramid, which has for its base the polygon, and for its vertex the point S. The triangle SMN, one of those which compose the convex surface of the pyramid, has for its measure the base MN multiplied by half of the altitude SA, which is at the same time the side of the given cone; this altitude being equal in all the triangles SNP, SPQ, &c., it follows that the convex surface of the pyramid is equal to the perimeter $MNPTM$ multiplied by $\tfrac{1}{2} SA$. But the perimeter $MNPTM$ is greater than *circ.* AO; therefore the convex surface of the pyramid is greater than *circ.* $AO \times \tfrac{1}{2} SA$, and consequently greater than the convex surface of the cone, which, with the same vertex S, has for its base the circle described with the radius OB. But on the contrary the convex surface of the cone is greater than that of the pyramid; for, if we apply the base of the pyramid to the base of an equal pyramid, and the base of the cone to that of an equal cone; the surface of the two cones will enclose on all sides the surface of the two pyramids; consequently the first surface will be greater than the second (514), and therefore the surface of the cone is greater than that of the pyramid, which is comprehended within it. The contrary would be the consequence of our hypothesis; accordingly this hypothesis cannot be maintained; therefore the circumference of the base of a cone multiplied by the half of its side cannot be the measure of the surface of a greater cone.

2. We say also, that this same product cannot be the measure of the surface of a less cone. For let *BO* be the radius of the base of the given cone, and, if it be possible, let *circ.* $BO \times \frac{1}{2} SB$ be the surface of a cone of which *S* is the vertex, and *AO*, less than *OB*, the radius of the base.

The same construction being supposed as above, the surface of the pyramid *SMNPT* will always be equal to the perimeter *MNPT* multiplied by $\frac{1}{2} SA$. Now the perimeter *MNPT* is less than *circ. BO*, and *SA* is less than *SB*; therefore for this double reason the convex surface of the pyramid is less than

$$circ.\ BO \times \tfrac{1}{2} SB,$$

which, by hypothesis, is the surface of a cone of which *AO* is the radius of the base; consequently the surface of the pyramid would be less than that of the inscribed cone. But on the contrary it is greater; for by applying the base of the pyramid to that of an equal pyramid, and the base of the cone to that of an equal cone, the surface of the two pyramids will enclose that of the two cones, and consequently will be greater. Therefore it is impossible that the circumference of the base of a given cone multiplied by the half of its side should be the measure of the surface of a less cone.

We conclude then, that the convex surface of a cone is equal to the circumference of the base multiplied by half of its side.

529. *Scholium.* Let *L* be the side of a cone, and *R* the radius of the base, the circumference of this base will be $2\pi R$, and the surface of the cone will have for its measure $2\pi R \times \frac{1}{2} L$, or πRL.

THEOREM.

530. *The convex surface of the frustum of a cone* ADEB (fig. 261) *is equal to its side* AD *multiplied by the half sum of the circumferences of the two bases* AB, DE. Fig. 261.

Demonstration. In the plane *SAB*, which passes through the axis *SO*, draw perpendicularly to *SA* the line *AF*, equal to the circumference which has for its radius *AO*; join *SF*, and draw *DH* parallel to *AF*.

On account of the similar triangles SAO, SDC,

$$AO : DC :: SA : SD;$$

and, on account of the similar triangles SAF, SDH,

$$AF : DH :: SA : SD;$$

whence $AF : DH :: AO : DC ::$ *circ.* AO : *circ.* DC (287). But, by construction, $AF =$ *circ.* AO; consequently

$$DH = \text{circ. } DC.$$

This being premised, the triangle SAF, which has for its measure $AF \times \frac{1}{2} SA$, is equal to the surface of a cone SAB, which has for its measure *circ.* $AO \times \frac{1}{2} SA$. For a similar reason the triangle SDH is equal to the surface of the cone SDE. Whence the surface of the frustum $ADEB$ is equal to that of the trapezoid $ADHF$. This has for its measure $AD \times \left(\frac{AF + DH}{2}\right)$ (178). Therefore the surface of the frustum of a cone $ADEB$ is equal to its side AD multiplied by the half sum of the circumferences of the two bases.

531. *Corollary.* Through the point I, the middle of AD, draw IKL parallel to AB, and IM parallel to AF; it may be shown as above that $IM =$ *circ.* IK. But the trapezoid (179)

$$ADHF = AD \times IM = AD \times \text{circ. } IK.$$

Hence we conclude further that *the surface of the frustum of a cone is equal to its side multiplied by the circumference of a section made at equal distances from the two bases.*

532. *Scholium.* If a line AD, situated entirely on the same side of the line OC and in the same plane, make a revolution about OC, the surface described by AD will have for its measure $AD \times \left(\frac{\text{circ. } AO + \text{circ. } DC}{2}\right)$, or $AD \times$ *circ.* IK; the lines AO, DC, IK, being perpendiculars let fall from the extremities and from the middle of the line AD upon the axis OC.

For, if we produce AD and OC till they meet in S, it is evident that the surface described by AD is that of the frustum of a cone, of which OA and DC are the radii of the bases, the entire cone having for its vertex the point S. Therefore this surface will have the measure stated.

This measure would always be correct, although the point D should fall upon S, which would give an entire cone, and also when the line AD is parallel to the axis, which would give a cylinder. In the first case DC would be nothing, in the second DC would be equal to AO and to IK.

LEMMA.

533. *Let* AB, BC, CD (fig. 262), *be several successive sides of a* Fig. 262. *regular polygon,* O *its centre, and* OI *the radius of the inscribed circle; if we suppose the portion of the polygon* ABCD, *situated entirely on the same side of the diameter* FG, *to make a revolution about this diameter, the surface described by* ABCD *will have for its measure* $MQ \times circ.\ OI$, MQ *being the altitude of this surface, or the part of the axis comprehended between the extreme perpendiculars* AM, DQ.

Demonstration. The point I being the middle of AB, and IK being a perpendicular to the axis let fall from the point I, the surface described by AB will have for its measure $AB \times circ.\ IK$ (532). Draw AX parallel to the axis, the triangles ABX, OIK, will have their sides perpendicular each to each, namely, OI to AB, IK to AX, and OK to BX; consequently these triangles will be similar, and will give the proportion

$$AB : AX \text{ or } MN :: OI : IK :: circ.\ OI : circ.\ IK,$$

therefore $AB \times circ.\ IK = MN \times circ.\ OI$. Whence it will be perceived that the surface described by AB is equal to its altitude MN multiplied by the circumference of the inscribed circle. Likewise the surface described by $BC = NP \times circ.\ OI$, the surface described by $CD = PQ \times circ.\ OI$. Accordingly the surface described by the portion of the polygon $ABCD$ has for its measure $(MN + NP + PQ) \times circ.\ OI$, or $MQ \times circ.\ OI$; therefore this surface is equal to its altitude multiplied by the circumference of the inscribed circle.

534. *Corollary.* If the entire polygon has an even number of sides, and the axis FG passes through two opposite vertices F and G, the entire surface described by the revolution of the semipolygon $FACG$ will be equal to its axis FG multiplied by the circumference of the inscribed circle. This axis FG will be at the same time the diameter of the circumscribed circle.

THEOREM.

535. *The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*

Demonstration. 1. We say that the diameter of a sphere multiplied by the circumference of a great circle cannot be the

measure of the surface of a greater sphere. For, if it be possi-
Fig. 263. ble, let $AB \times circ.\ AC$ (*fig.* 263) be the surface of a sphere whose radius is CD.

About the circle, whose radius is CA, circumscribe a regular polygon of an even number of sides, which shall not meet the circumference of the circle whose radius is CD; let M and S be two opposite vertices of this polygon; and about the diameter MS let the semipolygon MPS be made to revolve. The surface described by this polygon will have for its measure

$$MS \times circ.\ AC \quad (534);$$

but MS is greater than AB; therefore the surface described by the polygon is greater than $AB \times circ.\ AC$, and consequently greater than the surface of the sphere whose radius is CD. On the contrary the surface of the sphere is greater than the surface described by the polygon, since the first encloses the second on all sides. Therefore the diameter of a sphere multiplied by the circumference of a great circle cannot be the measure of the surface of a greater sphere.

2. We say also, that this same product cannot be the measure of the surface of a less sphere. For, if it be possible, let

$$DE \times circ.\ CD$$

be the surface of a sphere whose radius is CA. The same construction being supposed as in the first case, the surface of the solid generated by the polygon will always be equal to

$$MS \times circ.\ AC.$$

But MS is less than DE, and $circ.\ AC$ less than $circ.\ CD$; therefore for these two reasons the surface of the solid generated by the polygon would be less than $DE \times circ.\ CD$, and consequently less than the surface of the sphere whose radius is AC. But on the contrary the surface described by the polygon is greater than the surface of the sphere whose radius is AC, since the first surface encloses the second; therefore the diameter of a sphere multiplied by the circumference of a great circle cannot be the measure of the surface of a less sphere.

We conclude then, that the surface of a sphere is equal to the diameter multiplied by the circumference of a great circle.

536. *Corollary.* The surface of a great circle is measured by multiplying its circumference by half of the radius or a fourth of the diameter; therefore *the surface of a sphere is four times that of a great circle.*

537. ***Scholium.*** The surface of a sphere being thus measured and compared with plane surfaces, it will be easy to obtain the absolute value of lunary surfaces and spherical triangles, the ratio of which to the entire surface of the sphere has already been determined.

In the first place the lunary surface, whose angle is *A* (*fig.* 276), Fig. 276. is to the surface of the sphere, as the angle *A* is to four right angles (493), or as the arc of a great circle, which measures the angle *A*, is to the circumference of this same great circle. But the surface of the sphere is equal to this circumference multiplied by the diameter; therefore the lunary surface is equal to the arc, which measures the angle of this surface, multiplied by the diameter.

In the second place, every spherical triangle is equivalent to a lunary surface whose angle is equal to half of the excess of the sum of its three angles over two right angles (503). Let P, Q, R, be the arcs of a great circle which measure the three angles of a spherical triangle; let C be the circumference of a great circle and D its diameter; the spherical triangle will be equivalent to the lunary surface whose angle has for its measure $\frac{P+Q+R-\frac{1}{2}C}{2}$, and consequently its surface will be

$$D \times \frac{P+Q+R-\frac{1}{2}C}{2}.$$

Thus, in the case of the triangle of three right angles, each of the arcs P, Q, R, is equal to $\frac{1}{4}C$, and their sum is $\frac{3}{4}C$, the excess of this sum over $\frac{1}{2}C$ is $\frac{1}{4}C$, and the half of this excess is $\frac{1}{8}C$; therefore the surface of a triangle of three right angles $= \frac{1}{8}C \times D$ which is the eighth part of the whole surface of the sphere.

The measure of spherical polygons follows immediately from that of triangles, and it is moreover entirely determined by the proposition of art. 505, since the unit of measure, which is the triangle of three right angles, has just been estimated on a plane surface.

THEOREM.

538. *The surface of any spherical zone is equal to the altitude of this zone multiplied by the circumference of a great circle.*

Demonstration. Let EF (*fig.* 269) be any arc, either less or Fig. 269. greater than a quadrant, and let FG be drawn perpendicular to the radius EC; we say that the zone with one base, described by the revolution of the arc EF about EC, will have for its measure $EG \times circ.\ EC$.

For let us suppose, in the first place, that this zone has a less measure, and, if it be possible, let this measure be equal to $EG \times circ.\ AC$. Inscribe in the arc EF a portion of a regular polygon $EMNOPF$, the sides of which shall not touch the circumference described with the radius CA, and let fall upon EM the perpendicular CI, the surface described by the polygon EMF, turning about EC, will have for its measure $EG \times circ.\ CI$ (533). This quantity is greater than $EG \times circ.\ AC$, which, by hypothesis, is the measure of the zone described by the arc EF. Consequently the surface described by the polygon $EMNOPF$ would be greater than the surface described by the circumscribed arc EF; but on the contrary this last surface is greater than the first, since it encloses it on all sides; therefore the measure of any spherical zone with one base cannot be less than the altitude of this zone multiplied by the circumference of a great circle.

We say, in the second place, that the measure of the same zone cannot be greater than the altitude of this zone multiplied by the circumference of a great circle. For, let us suppose that the zone in question is the one described by the arc AB about AC, and, if it be possible, let the *zone* AB be greater than

$$AD \times circ.\ AC.$$

The entire surface of the sphere composed of the two *zones* AB, BH, has for its measure $AH \times circ.\ AC$ (535), or

$$AD \times circ.\ AC + DH \times circ.\ AC;$$

if then the *zone* AB be greater than $AD \times circ.\ AC$, the *zone* BH must be less than $DH \times circ.\ AD$, which is contrary to the first part already demonstrated. Therefore the measure of a spherical zone with one base cannot be greater than the altitude of this zone multiplied by the circumference of a great circle.

It follows then that every spherical zone with one base has for its measure the altitude of this zone multiplied by the circumference of a great circle.

Let us now consider any zone of two bases described by the Fig. 220. revolution of the arc FH (*fig.* 220) about the diameter DE, and let FO, HQ, be drawn perpendicular to this diameter. The zone described by the arc FH is the difference of the two zones described by the arcs *DH* and *DF*; these have for their measure respectively *DQ* × *circ.* *CD* and *DO* × *circ.* *CD*; therefore the zone described by *FH* has for its measure

$$(DQ - DO) \times circ.\ CD \text{ or } OQ \times circ.\ CD.$$

We conclude then that every spherical zone with one or two bases has for its measure the altitude of this zone multiplied by the circumference of a great circle.

539. *Corollary.* Two zones are to each other as their altitudes, and any zone whatever is to the surface of the sphere as the altitude of this zone is to the diameter.

THEOREM.

540. *If the triangle* BAC (fig. 264, 265) *and the rectangle* BCEF *of the same base and same altitude turn simultaneously about the common base* BC, *the solid generated by the revolution of the triangle will be a third of the cylinder generated by the revolution of the rectangle.* Fig. 264, 265.

Demonstration. Let fall upon the axis the perpendicular *AD* (*fig.* 264); the cone generated by the triangle *ABD* is a third of the cylinder generated by the rectangle *AFBD* (524); also the cone generated by the triangle *ADC* is a third of the cylinder generated by the rectangle *ADCE*; therefore the sum of the two cones, or the solid generated by *ABC*, is a third of the sum of the two cylinders, or of the cylinder generated by the rectangle *BCEF*. Fig. 264.

If the perpendicular *AD* (*fig.* 265) fall without the triangle, the solid generated by *ABC* will be the difference of the cones generated by *ABD* and *ACD*; but, at the same time, the cylinder generated by *BCEF* will be the difference of the cylinders generated by *AFBD*, *AECD*. Therefore the solid generated by the revolution of the triangle will be always the third of the cylinder generated by the revolution of the rectangle of the same base and same altitude. Fig. 165.

541. *Scholium.* The circle of which *AD* is the radius has for its surface $\pi \times \overline{AD}^2$; consequently $\pi \times \overline{AD}^2 \times BC$ is the measure of the cylinder generated by *BCEF*, and $\frac{1}{3} \pi \times \overline{AD}^2 \times BC$ is the measure of the solid generated by the triangle *ABC*.

PROBLEM.

Fig. 266. 542. *The triangle* CAB (fig. 266) *being supposed to make a revolution about the line* CD, *drawn at pleasure without the triangle through the vertex* C, *to find the measure of the solid thus generated.*

Solution. Produce the side AB until it meet the axis CD in D, and from the points A, B, let fall upon the axis the perpendiculars AM, BN.

The solid generated by the triangle CAD has for its measure $\frac{1}{3}\pi \times \overline{AM}^2 \times CD$ (540); the solid generated by the triangle CBD has for its measure $\frac{1}{3}\pi \overline{BN}^2 \times CD$; therefore the difference of these solids, or the solid generated by ABC, will have for its measure $\frac{1}{3}\pi \times (\overline{AM}^2 - \overline{BN}^2) \times CD$.

This expression will admit of another form. From the point I, the middle of AB, draw IK perpendicular to CD, and through the point B draw BO parallel to CD, we shall have

$$AM + BN = 2IK \quad (178),$$

and $AM - BN = AO$; consequently $(AM + BN) \times (AM - BN)$, or $\overline{AM}^2 - \overline{BN}$ (184), is equal to $2IK \times AO$. Accordingly the measure of the solid under consideration will also be expressed by $\frac{2}{3}\pi \times IK \times AO \times CD$. But, if the perpendicular CP be let fall upon AB, the triangles ABO, DCP will be similar, and will give the proportion $AO : CP :: AB : CD$; whence

$$AO \times CD = CP \times AB;$$

moreover $CP \times AB$ is double of the area of the triangle ABC; thus we have $AO \times CD = 2ABC$; consequently the solid generated by the triangle ABC has also for its measure

$$\frac{4}{3}\pi \times ABC \times KI,$$

or, since *circ.* KI is equal to $2\pi \times KI$, this same measure will be $ABC \times \frac{2}{3}$ *circ.* KI. Therefore, *the solid generated by the revolution of the triangle* ABC *has for its measure the area of this triangle multiplied by two thirds of the circumference described by the point* I *the middle of the base.*

Fig. 267. 543. *Corollary.* If the side $AC = CB$ (*fig.* 267), the line CI will be perpendicular to AB, the area ABC will be equal to $AB \times \frac{1}{2} CI$, and the solidity $\frac{4}{3}\pi \times ABC \times IK$ will become $\frac{2}{3}\pi \times AB \times IK \times CI$. But the triangles ABO, CIK, are similar and give the proportion $AB : BO$ or $MN :: CI : IK$; consequently

$$AB \times IK = MN \times CI;$$

therefore the solid generated by the isosceles triangle ABC will have for its measure $\frac{2}{3}\pi \times MN \times \overline{CI}^2$.

544. *Scholium.* The general solution seems to suppose that the line AB produced would meet the axis, but the results would not be the less true, if the line AB were parallel to the axis.

Indeed the cylinder generated by $AMNB$ (*fig.* 268) has for its measure $\pi \times \overline{AM}^2 \times MN$, the cone generated by ACM is equal to Fig. 268.

$$\tfrac{1}{3}\pi \times \overline{AM}^2 \times CM,$$

and the cone generated by $BCN = \frac{1}{3}\pi \times \overline{AM}^2 \times CN$. Adding the two first solids together and subtracting the third from the sum, we have for the solid generated by ABC

$$\pi \times \overline{AM}^2 \times (MN + \tfrac{1}{3}CM - \tfrac{1}{3}CN);$$

and, since $\frac{1}{3}CM - \frac{1}{3}CN = -(\frac{1}{3}CN - \frac{1}{3}CM) = -\frac{1}{3}MN$, the above expression reduces itself to $\pi \times \overline{AM}^2 \times \frac{2}{3}MN$, or

$$\tfrac{2}{3}\pi \times \overline{CP}^2 \times MN,$$

which agrees with the results already found.

THEOREM.

545. *Let* AB, BC, CD (fig. 262), *be several successive sides of a regular polygon,* O *its centre,* OI *the radius of the inscribed circle; if we suppose the polygonal sector* AOD, *situated on the same side of the diameter* FG, *to make a revolution about this diameter, the solid generated will have for its measure* $\frac{2}{3}\pi \times \overline{\mathrm{OI}}^2 \times \mathrm{MQ}$, MQ *being the portion of the axis terminated by the extreme perpendiculars* AM, DQ. Fig. 262.

Demonstration. Since the polygon is regular, all the triangles AOB, BOC, &c., are equal and isosceles. Now, by the corollary of the preceding proposition, the solid generated by the isosceles triangle AOB has for its measure $\frac{2}{3}\pi \times \overline{OI}^2 \times MN$, the solid generated by the triangle BOC has for its measure $\frac{2}{3}\pi \times \overline{OI}^2 \times NP$, and the solid generated by the triangle COD has for its measure $\frac{2}{3}\pi \times \overline{OI}^2 \times PQ$; therefore the sum of these solids, or the entire solid generated by the polygonal sector AOD, has for its measure $\frac{2}{3}\pi \times \overline{OI}^2(MN + NP + PQ)$, or $\frac{2}{3}\pi \times \overline{OI}^2 \times MQ$.

THEOREM.

546. *Every spherical sector has for its measure the zone which serves as a base multiplied by a third of the radius, and the entire sphere has for its measure its surface multiplied by a third of the radius.*

Fig. 269. *Demonstration.* Let ABC (*fig.* 269) be the circular sector, which, by its revolution about AC, generates the spherical sector; the zone described by AB being $AD \times$ *circ.* AC, or

$$2\pi \times AC \times AD \quad (538),$$

we say that the spherical sector will have for its measure this zone multiplied by $\frac{1}{3} AC$, or $\frac{2}{3} \pi \times \overline{AC}^2 \times AD$.

1. Let us suppose, if it be possible, that this quantity

$$\tfrac{2}{3} \pi \times \overline{AC}^2 \times AD$$

is the measure of a greater spherical sector, of the spherical sector, for example, generated by the circular sector ECF similar to ACB.

Inscribe in the arc EF a portion of a regular polygon $EMNF$ the sides of which shall not meet the arc AB, then suppose the polygonal sector $ENFC$ to turn about EC at the same time with the circular sector ECF. Let CI be the radius of a circle inscribed in the polygon, and let FG be drawn perpendicular to EC. The solid generated by the polygonal sector will have for its measure $\frac{2}{3} \pi \times \overline{CI}^2 \times EG$ (545); now CI is greater than AC, by construction, and EG is greater than AD; for, if we join AB, EF, the triangles EFG, ABD, which are similar, give the proportion $EG : AD :: FG : BD :: CF : CB$; therefore $EG > AD$.

For this double reason $\frac{2}{3} \pi \times \overline{CI}^2 \times EG$ greater than

$$\tfrac{2}{3} \pi \times \overline{CA}^2 \times AD;$$

the first expression is the measure of the solid generated by the polygonal sector, the second is, by hypothesis, that of the spherical sector generated by the circular sector ECF; consequently the solid generated by the polygonal sector would be greater than the spherical sector generated by the circular sector. But on the contrary the solid in question is less than the spherical sector, since it is contained in it; accordingly the hypothesis with which we set out cannot be maintained; therefore the zone or base of a spherical sector multiplied by a third of the radius cannot be the measure of a greater spherical sector.

2. We say that this same product cannot be the measure of a less spherical sector. For, let CEF be the circular sector which by its revolution generates the given spherical sector, and let us suppose, if it be possible, that $\frac{2}{3}\pi \times \overline{CE}^2 \times EG$ is the measure of a less spherical sector, of that, for example, generated by the circular sector ACB.

The preceding construction remaining the same, the solid generated by the polygonal sector will always have for its measure $\frac{2}{3}\pi \times \overline{CI}^2 \times EG$. But CI is less than CE; consequently the solid is less than $\frac{2}{3}\pi \times \overline{CE}^2 \times EG$, which, by hypothesis, is the measure of the spherical sector generated by the circular sector ACB. Therefore the solid generated by the polygonal sector would be less than the solid generated by the spherical sector; but on the contrary it is greater, since it contains it. Therefore it is impossible that the zone of a spherical sector multiplied by a third of the radius should be the measure of a less spherical sector.

We conclude then, that every spherical sector has for its measure the zone which answers as a base multiplied by a third of the radius.

A circular sector ACB may be increased till it becomes equal to a semicircle; then the spherical sector generated by its revolution is an entire sphere. Therefore *the solidity of a sphere is equal to its surface multiplied by a third of the radius.*

547. *Corollary.* The surfaces of spheres being as the squares of their radii, these surfaces multiplied by the radii are as the cubes of the radii. Therefore *the solidities of two spheres are as the cubes of their radii*, or *as the cubes of their diameters.*

548. *Scholium.* Let R be the radius of a sphere, its surface will be $4\pi R^2$, and its solidity $4\pi R^2 \times \frac{1}{3}R$, or $\frac{4}{3}\pi R^3$. If we call D the diameter, we shall have $R = \frac{1}{2}D$, and $R^3 = \frac{1}{8}D^3$; therefore the solidity will also be expressed by $\frac{4}{3}\pi \times \frac{1}{8}D^3$, or $\frac{1}{6}\pi D^3$.

THEOREM.

549. *The surface of a sphere is to the whole surface of the circumscribed cylinder, the bases being comprehended, as 2 is to 3; and the solidities of these two bodies are in the same ratio.*

Fig. 270. *Demonstration.* Let *MPNQ* (*fig.* 270) be a great circle of the sphere, *ABCD* the circumscribed square; if the semicircle *PMQ*, and the semisquare *PADQ*, be made to turn at the same time about the diameter *PQ*, the semicircle will generate the sphere, and the semisquare will generate the cylinder circumscribed about the sphere.

The altitude *AD* of the cylinder is equal to the diameter *PQ*, the base of the cylinder is equal to a great circle, since it has for a diameter *AB* equal to *MN*; consequently the convex surface of the cylinder is equal to the circumference of a great circle multiplied by its diameter (523). This measure is the same as that of the surface of the sphere (535); whence it follows *that the surface of the sphere is equal to the convex surface of the circumscribed cylinder.*

But the surface of the sphere is equal to four great circles; consequently the convex surface of the circumscribed cylinder is also equal to four great circles. If we add the two bases, which are equal to two great circles, the whole surface of the circumscribed cylinder will be equal to six great circles; therefore the surface of the sphere is to the whole surface of the circumscribed cylinder as 4 is to 6, or as 2 is to 3. This is the first part of the proposition which it was proposed to demonstrate.

In the second place, since the base of the circumscribed cylinder is equal to a great circle, and its altitude equal to the diameter, the solidity of the cylinder will be equal to a great circle multiplied by the diameter (516). But the solidity of the sphere is equal to four great circles multiplied by a third of the radius (546), which amounts to a great circle multiplied by $\frac{4}{3}$ of the radius, or $\frac{2}{3}$ of the diameter; therefore the sphere is to the circumscribed cylinder as 2 is to 3, and consequently the solidities of these two bodies are to each other as their surfaces.

550. *Scholium.* If a polyedron be supposed, all whose faces touch the sphere, this polyedron might be considered as composed of pyramids, having the centre of the sphere for their common vertex, the bases being the several faces of the polye-

dron. Now it is evident that all these pyramids will have for their common altitude the radius of the sphere, so that each pyramid will be equal to a face of the polyedron, which serves as a base, multiplied by a third of the radius; therefore the entire polyedron will be equal to its surface multiplied by a third of the radius of the inscribed sphere.

It will be perceived by this that the solidities of polyedrons circumscribed about a sphere are to each other as the surfaces of these same polyedrons. Thus the property which we have demonstrated for the circumscribed cylinder is common to an infinite number of other bodies.

We might have remarked also that the surfaces of polygons circumscribed about a circle are to each other as their perimeters.

PROBLEM.

551. *The circular segment* BMD (fig. 271) *being supposed to revolve about a diameter exterior to this segment, to find the value of the solid generated.* Fig. 271.

Solution. Let fall upon the axis the perpendiculars *BE*, *DF*, and upon the chord *BD* the perpendicular *CI*, and draw the radii *CB*, *CD*.

The solid generated by the sector $BCA = \frac{2}{3}\pi \times \overline{CB}^2 \times AE$ (546); the solid generated by the sector $DCA = \frac{2}{3}\pi \times \overline{CB}^2 \times AF$; consequently the difference of these two solids, or the solid generated by the sector *DCB*, will be equal to

$$\tfrac{2}{3}\pi \times \overline{CB}^2 \times (AF - AE) = \tfrac{2}{3}\pi \times \overline{CB}^2 \times EF.$$

But the solid generated by the isosceles triangles *DCB* has for its measure $\frac{2}{3}\pi \times \overline{CI}^2 \times EF$ (543); consequently the solid generated by the segment $BMD = \frac{2}{3}\pi \times EF \times \left(\overline{CB}^2 - \overline{CI}^2\right)$. Now in the right-angled triangle *CBI* we have $\overline{CB}^2 - \overline{CI}^2 = \overline{BI}^2 = \frac{1}{4}\overline{BD}^2$; therefore the solid generated by the segment *BMD* has for its measure $\frac{2}{3}\pi \times EF \times \frac{1}{4}\overline{BD}^2$, or $\frac{1}{6}\pi\overline{BD}^2 \times EF$.

552. *Scholium.* The solid generated by the segment *BMD* is to the sphere whose diameter is *BD*, as $\frac{1}{6}\pi \times \overline{BD}^2 \times EF$ is to $\frac{1}{6}\pi \times \overline{BD}^3$, or :: $EF : BD$.

THEOREM.

553. *Every segment of a sphere, comprehended between two parallel planes, has for its measure the half sum of its bases multiplied by its altitude, plus the solidity of the sphere of which this same altitude is the diameter.*

Fig. 271. *Demonstration.* Let BE, DF (*fig.* 271), be the radii of the bases of the segment, EF its altitude, so that the segment may be formed by the revolution of the circular space $BMDFE$ about the axis FE. The solid generated by the segment BMD will be equal to $\frac{1}{6}\pi \times \overline{BD}^2 \times EF$ (552), the frustum of a cone generated by the trapezoid $BDFE$ will be equal to

$$\tfrac{1}{3}\pi \times EF \times (\overline{BE}^2 + \overline{DF}^2 + BE \times DF) \quad (527);$$

consequently the segment of the sphere which is sum of these two solids $= \frac{1}{6}\pi \times EF \times (2\overline{BE}^2 + 2\overline{DF}^2 + 2BE \times DF + \overline{BD}^2.)$ But, by drawing BO parallel to EF, we shall have $DO = DF - BE$, $\overline{DO}^2 = \overline{DF}^2 - 2DF \times BE + \overline{BE}^2$ (182), and consequently $\overline{BD}^2 = \overline{BO}^2 + \overline{DO}^2 = \overline{EF}^2 + \overline{DF}^2 - 2DF \times BE + \overline{BE}^2$. Putting this value in the place of $\overline{BD}^2$ in the expression for the segment, and reducing it, we shall have for the solidity of the segment

$$\tfrac{1}{6}\pi \times EF \times (3\overline{BE}^2 + 3\overline{DF}^2 + \overline{EF}^2),$$

an expression which may be decomposed into two parts; the one $\frac{1}{6}\pi \times EF \times (3\overline{BE}^2 + 3\overline{DF}^2)$, or $EF \times \left(\frac{\pi \times \overline{BE}^2 + \pi \times \overline{DF}^2}{2}\right)$, is the half sum of the bases multiplied by the altitude; the other $\frac{1}{6}\pi \times \overline{EF}^3$ represents the sphere of which EF is the diameter (548); therefore the segment of the sphere &c.

554. *Corollary.* If one of the bases is nothing, the segment in question becomes a spherical segment having only one base; therefore *every spherical segment having only one base is equivalent to half of the cylinder of the same base and same altitude, plus the sphere of which this altitude is the diameter.*

General Scholium.

555. Let R be the radius of the base of a cylinder, H its altitude; the solidity of the cylinder will be $\pi R^2 \times H$, or $\pi R^2 H$.

Let R be the radius of the base of a cone, H its altitude; the solidity of the cone will be $\pi R^2 \times \frac{1}{3} H$, or $\frac{1}{3} \pi R^2 H$.

Let A, B, be the radii of the bases of the frustum of a cone, H its altitude, the solidity of the frustum will be

$$\tfrac{1}{3} \pi H (A^2 + B^2 + AB).$$

Let R be the radius of a sphere; its solidity will be $\frac{4}{3} \pi R^3$.

Let R be the radius of a spherical sector, H the altitude of the zone, which answers as a base; the solidity of the sector will be $\frac{2}{3} \pi R^2 H$.

Let P, Q, be the two bases of a spherical segment, H its altitude, the solidity of this segment will be $\left(\frac{P + Q}{2}\right) \times H + \frac{1}{6} \pi H^3$.

If the spherical segment have only one base P, its solidity will be $\frac{1}{2} PH + \frac{1}{6} \pi H^3$.

Appendix to the Third Section of the Second Part.

OF SPHERICAL ISOPERIMETRICAL POLYGONS.

THEOREM.

556. *Let* S *be the number of solid angles in a polyedron,* H *the number of its faces,* A *the number of its edges; then in all cases we shall have* $S + H = A + 2$.

Demonstration. Within the polyedron, take a point, from which let straight lines be drawn to the vertices of all its angles; conceive next, that from the same point as a centre, a spherical surface is described, meeting all these straight lines in as many points; join these points by arcs of great circles, so as to form on the surface of the sphere polygons corresponding in position and number with the faces of the polyedron. Let $ABCDE$ be one of these polygons (*fig.* 240), and n the number of its sides; its surface will be $s - 2n + 4$, s being the sum of the angles A, B, C, D, E (506). If the surface of each polygon be estimated in a similar manner, and afterwards the whole added together, we shall find their sum, or the surface of the sphere, represented by 8, to be equal to the sum of all the angles of the Fig. 24

polygons, minus twice the number of their sides, plus 4, taken as many times as there are faces. Now, since all the angles which meet at any one point A are equal to four right angles, the sum of all the angles of the polygons must be equal to 4, taken as many times as there are solid angles; it is therefore equal to $4S$. Also, twice the number of sides AB, BC, CD, &c., is equal to four times the number of edges, or to $4A$; since the same edge is in every case a side to two faces. Hence we have $8 = 4S - 4A + 4H$; or, dividing the whole by 4, $2 = S - A + H$; therefore $S + H = A + 2$.

557. *Corollary.* From this it follows, that *the sum of all the plane angles, which form the solid angles of a polyedron, is equal to as many times four right angles as there are units in* S — 2, S *being the number of solid angles of the polyedron.*

For, if we consider a face, the number of whose sides is n, we shall find that the sum of the angles of this face is equal to $2n - 4$ right angles (79). But the sum of all these $2n$'s, or twice the number of sides in all the faces, will be $4A$; and 4, taken as many times as there are faces, will be $4H$; hence the sum of the angles in all the faces is $4A - 4H$. Now, by the theorem just demonstrated, we have $A - H = S - 2$, and consequently $4A - 4H = 4(S - 2)$. Therefore *the sum of all the plane angles, &c.*

THEOREM.

558. *Of all the spherical triangles formed with two given sides* CA, CB (fig. 276), *and a third assumed at pleasure, the greatest,* ABC, *is that in which the angle* C, *contained by the given sides, is equal to the sum of the two other angles,* A *and* B. Fig. 276.

Demonstration. Produce the two sides AC, AB, till they meet in D; we shall have a spherical triangle BCD, in which the angle DBC is also equal to the sum of the two other angles BDC, BCD. For, $BCD + BCA$, being equal to two right angles, and likewise $CBA + CBD$, we have

$$BCD + BCA = CBA + CBD;$$

and adding $BDC = BAC$, we shall have

$$BCD + BCA + BDC = CBA + CBD + BAC.$$

Now, by hypothesis, $BCA = CBA + BAC$; hence

$$CBD = BCD + BDC.$$

Draw BI making the angle $CBI = BCD$, and consequently $IBD = BDC$; the two triangles IBD, IBC, will be isosceles, and we shall have $IC = IB = ID$. Hence the point I, the middle point of DC, is at equal distances from the three points B, C, D. For a similar reason, the point O, the middle of BA, is equally distant from the points A, B, C.

Now, suppose $CA = CA$ and the angle $BCA' > BCA$; if $A'B$ be joined, and the arcs $A'C$, $A'B$, produced till they meet in D', the arc $D'CA'$ will be a semicircumference, as well as DCA; therefore, since we have $CA' = CA$, we shall also have $CD' = CD$. But in the triangle CID', we have $CI + ID' > CD'$; hence $ID' > CD - CI$, or $ID' > ID$.

In the isosceles triangle CIB, bisect the angle I by the arc EIF, which will also bisect BC at right angles. If a point L is assumed between I and E, the distance BL, equal to LC, will be less than BI; for it might be shewn as in art. 41, that $BL + LC < BI + IC$; and, taking the half of each, that $BL < BI$. But in the triangle $D'LC$, we have $D'L > D'C - CL$, and still more $D'L > DC - CI$, or $D'L > DI$, or $D'L > BI$; cnosequently $D'L > BL$. Hence, if in the arc EIF, we seek for a point equally distant from the three points B, C, D', it can be found only in the prolongation of EI towards F. Let I' be the point required; we shall have $D'I' = BI' = CI'$; the triangles $I'CB$, $I'CD'$, $I'BD'$, being isosceles, we shall have the equal angles $I'BC = I'CB$, $I'BD' = I'D'B$, $I'CD' = I'D'C$. But the angles $D'BC + CBA'$ are equal to two right angles, and

$$D'CB + BCA'$$

are likewise equal to two right angles; therefore

$$D'BI' + I'BC + CBA' = 2,$$

$$BCI' - I'CD' + BCA' = 2.$$

Add together the two equations observing that $I'BC = BCI'$, and $D'BI' - I'CD' = BD'I' - I'D'C = CD'B = CA'B$; and we shall have $\quad 2\,I'BC + CA'B + CBA' + BCA' = 4.$

Hence $CA'B + CBA' + BCA' - 2$ (which measures the area of the triangle $A'BC$ (501) $= 2 - 2\,I'BC$; so that we have

$$\textit{area } A'BC = 2 - 2 \textit{ angle } I'BC;$$

likewise, in the triangle ABC, we should have

$$\textit{area } ABC = 2 - 2 \textit{ angle } IBC.$$

Now the angle $I'BC$ has already been proved to be greater than IBC; hence the area $A'BC$ is less than ABC.

The same demonstration would lead to the same conclusion, Fig. 277. if, taking always the arc $CA' = CA$, the angle BCA' (*fig.* 277) were made less than BCA; hence ABC is the greatest of all those triangles, which have two sides given, and the third to be assumed at pleasure.

Fig. 278. 559. *Scholium* I. The triangle ABC (*fig.* 278), the greatest of all those which have two given sides CA, CB, may be inscribed in a semicircle, the diameter of which is the chord of the third side AB; for O being the middle point of AB, the distances OC, OB, as we have seen, are equal; hence the circumference of a small circle, described from the point O as a pole, with the distance OB, will pass through the three points A, B, C. Moreover, the straight line AB is a diameter to this small circle; for the centre, which must be at once in the plane of the small circle, and (456) in the plane of the arc of the great circle BOA, must of necessity be found in the intersection of those two planes, which is the straight line BA; hence BA will be a diameter.

560. *Scholium* II. In the triangle ABC, the angle C being equal to the sum of the other two A and B, the sum of all the three angles must be double of the angle C. But (489) that sum is always greater than two right angles; hence C is always greater than one.

561. *Scholium* III. If the sides CB, CA, are produced till they meet in E, the triangle BAE will be equal to the fourth part of the surface of the sphere. For the angle

$$E = C = ABC + CAB;$$

hence the three angles of the triangle BAE are equivalent to the four ABC, ABE, CAB, BAE, whose sum is equal to four right angles; therefore (505) the surface of the triangle

$$BAE = 4 - 2 = 2,$$

which is the fourth part of the surface of the sphere.

562. *Scholium* IV. There could be no *maximum*, if the sum of the two given sides CA, CB, were equal to, or greater than, the semicircumference of a great circle. For, since the triangle ABC must be capable of being inscribed in a semicircle of the sphere, the sum of the two sides CA, CB, will be less (460) than the semicircumference BCA, and consequently less than half the circumference of a great circle.

The reason why there can be no *maximum*, when the sum of the two given sides is greater than the semicircumference of a great circle, is that in this case the triangle continues to augment, as the angle contained by its two given sides augments; and at last, when this angle becomes equal to two right angles, the three sides are all in the same plane, and form a whole circumference; the spherical triangle has then increased to a hemisphere, but it has at the same time ceased to be a triangle.

THEOREM.

563. *Of all the spherical triangles, formed with a given side and a given perimeter, the greatest is that in which the two undetermined sides are equal.*

Demonstration. Let AB (*fig.* 279) the given side be common to the two triangles ACB, ADB, and let $AC + CB = AD + DB$; we are to show that the isosceles triangle ACB, in which $AC = CB$, is greater than ADB, which is not isosceles. Fig. 279.

Since these triangles have the common part AOB, it will be sufficient to prove that the triangle BOD is less than AOC. Now, the angle CBA, equal to CAB, is greater than OAB; therefore (497) the side AO is greater than OB. Take $OI = OB$, make $OK = OD$, and join KI; the triangle OKI (497) will be equal to DOB. Now, if the triangle DOB, or its equal KOI, is not admitted to be less than OAC, it must be either equal or greater; in both which cases, since the point I is between A and O, the point K must be found in the prolongation of OC, otherwise the triangle OKI would be contained in the triangle CAO, and therefore less than CAO. This granted, since the shortest way from C to A is CA, we have $CK + KI + IA > CA$. But

$$CK = OD - CO,\ AI = AO - OB,\ KI = BD;$$

hence $OD - CO + AO - OB + BD > CA$, or by reduction, $AD - CB + BD > CA$, or $AD + BD > CA + CB$. But this inequality is at variance with the supposition of

$$AD + BD = CA + CB;$$

hence the point K cannot fall in the prolongation of OC; consequently it falls between O and C, and the triangle KOI or its equal ODB is less than ACO; therefore the isosceles triangle ACB is greater than ADB having the same base and perimeter, which is not isosceles.

564. *Scholium.* The two last theorems are analogous to those of art. 63 and 69, of the appendix to section fourth; and from

them may be deduced, with regard to spherical polygons, the same consequences as we have obtained respecting plane polygons. The chief are as follows:

565. *Among spherical polygons of the same perimeter and the same number of sides, that is the greatest which has its sides equal.*

The demonstration is the same as that of art. 301.

566. *Among spherical polygons formed of known sides and one side taken at pleasure, the greatest is that which can be inscribed in a semicircle the diameter of which is equal to the chord of the undetermined side.*

The demonstration is deduced from art. 559, in the manner exhibited in art. 303. It is requisite for the existence of a *maximum*, that the sum of the given sides be less than the semicircumference of a great circle.

567. *Among spherical polygons formed of given sides, the greatest is that which can be inscribed in a circle of the sphere.*

The demonstration is the same as that of art. 303.

568. *Among spherical polygons which have the same perimeter and the same number of sides, the greatest is that which has its angles equal, and its sides equal.*

This results from the first and the third of the above propositions.

Note. All the propositions relating to the *maxima* of spherical polygons, are also applicable to solid angles, of which these polygons are the measures.

Appendix to Sections Second and Third.

OF THE REGULAR POLYEDRONS.

THEOREM.

569. *There can be only five regular polyedrons.*

Demonstration. For, *regular polyedrons* were defined as having equal regular polygons for their faces, and all their solid angles equal. These conditions cannot be fulfilled except in a small number of cases.

1. If the faces are equilateral triangles, polyedrons may be formed of them, having solid angles contained by three of those triangles, by four, or by five: hence arise three regular bodies, the *tetraedron*, the *octaedron*, the *icosaedron*. No other can be formed with equilateral triangles; for six angles of such a triangle are equal to four right angles, and (356) cannot form a solid angle.

2. If the faces are squares, their angles may be arranged by threes: hence results the *hexaedron* or *cube*. Four angles of a square are equal to four right angles, and cannot form a solid angle.

3. In fine, if the faces are regular pentagons, their angles may likewise be arranged by threes; the regular *dodecaedron* will thus be formed.

We can proceed no farther; three angles of a regular hexagon are equal to four right angles; three of a heptagon are greater.

Hence there can be only five regular polyedrons; three formed with equilateral triangles, one with squares, and one with pentagons.

570. *Scholium.* In the following problem, we shall show that these five polyedrons actually exist; and that all their dimensions may be determined, when one of their faces is known.

PROBLEM.

571. *One of the faces of a regular polyedron, or only a side of it, being given, to construct the polyedron.*

Solution. This problem admits of five cases, which we proceed to solve in succession.

Construction of the Tetraedron.

572. Let ABC (*fig.* 280) be the equilateral triangle which is to form one of the faces of the tetraedron. At the point O, the centre of this triangle, erect OS perpendicular to the plane ABC; let this perpendicular terminate in S, so that $AS = AB$; join SB, SC; the pyramid S-ABC will be the tetraedron required. Fig. 280.

For, on account of the equal distances OA, OB, OC, the oblique lines SA, SB, SC are equally removed from the perpendicular SO, and consequently equal to each other. One of them $SA = AB$; hence the four faces of the pyramid S-ABC are triangles, equal to the given triangle *ABC*. And the solid angles of this pyramid are all equal, because each of them is formed by three equal plane angles: this pyramid therefore is a regular tetraedron.

Construction of the Hexaedron.

Fig. 281. 573. Let *ABCD* (*fig.* 281) be a given square. On the base *ABCD*, construct a right prism whose altitude *AE* shall be equal to the side *AB*. The faces of this prism will evidently be equal squares; and its solid angles all equal to each other, each being formed by three right angles; this prism therefore is a regular hexaedron or cube.

Construction of the Octaedron.

Fig. 282. 574. Let *AMB* (*fig.* 282) be a given equilateral triangle. On the side *AB*, describe a square *ABCD*; through the point *O*, the centre of this square, let the perpendicular *TS* be drawn, terminating on the one hand and on the other in *T* and *S*, so that $OT = OS = OA$; then join *SA*, *SB*, *TA*, &c.; we shall have a solid *SABCDT*, composed of two quadrangular pyramids *S-ABCD*, *T-ABCD*, united together by their common base *ABCD*; this solid will be the required octaedron.

For, the triangle *AOS* is right-angled at *O*, and likewise the triangle *AOD*; the sides *AO*, *OS*, *OD*, are equal to each other; hence those triangles are equal, and $AS = AD$. In the same manner we could shew, that, all the other right-angled triangles *AOT*, *BOS*, *COT*, &c., are equal each to the triangle *AOD*; hence all the sides *AB*, *AS*, *AT*, &c., are equal to each other, and therefore the solid *SABCDT* is contained by eight triangles, each equal to the given equilateral triangle *ABM*. We have yet to shew that the solid angles of this polyedron are equal to each other; that the angle *S*, for example, is equal to the angle *B*.

Now, the triangle *SAC* is evidently equal to the triangle *DAC*, and therefore the angle *ASC* is a right angle; hence the figure *SATC* is a square equal to the square *ABCD*. But if we compare the pyramid *B-ASCT* with the pyramid *S-ABCD*, we shall see that the base *ASCT* of the first may be placed on the base *ABCD* of the second; then, the point *O* being their common centre, the altitude *OB* of the first will coincide with the altitude

OS of the second; and the two pyramids will exactly coincide with each other in all points; hence the solid angle *S* is equal to the solid angle *B*; and therefore the solid *SABCDT* is a regular octaedron.

575. *Scholium.* If three equal straight lines *AC*, *BD*, *ST*, are perpendicular to each other, and bisect each other, the extremities of these straight lines will be the vertices of a regular octaedron.

Construction of the Dodecaedron.

576. Let *ABCDE* (*fig.* 283) be a given regular pentagon; let *ABP*, *CBP*, be two plane angles each equal to the angle *ABC*. With these plane angles form the solid angle *B*; and by art. 361 determine the mutual inclination of two of these planes; which inclination we shall call *K*. In like manner, at the points *C*, *D*, *E*, *A*, form solid angles, equal to the solid angle *B*, and which shall be similarly situated; the plane *CBP* will be the same as the plane *BCG*, since both of them are inclined at an equal angle *K* to the plane *ABCD*; hence in the plane *PBCG*, we may describe the pentagon *BCGFP*, equal to the pentagon *ABCDE*. If the same thing is done in each of the other planes *CDI*, *DEL*, &c., we shall have a convex surface *PEGH*, &c., composed of six regular pentagons, all equal to each other, and each inclined to its adjacent plane by the same quantity *K*. Let *pfgh*, &c. be a second surface equal to *PFGH*, &c.; we say that these two surfaces may be joined so as to form only a single continuous convex surface. For the angle *opf*, for example, may be joined to the two angles *OPB*, *BPF*, so as to make a solid angle *P* equal to the angle *B*; and by this joining together no change will take place in the inclination of the planes *BPF*, *BPO*, that inclination being already such as is required to form the solid angle. But whilst the solid angle *P* is forming, the side *pf* will apply itself to its equal *PF*, and at the point *F* will be found three plane angles *PFG*, *pfe*, *efg*, united so as to form a solid angle equal to each of the solid angles already formed; and this junction, like the former, will take place without producing any change either in the state of the angle *P* or in that of the surface *efgh*, &c.; for the planes *PFG*, *efp*, already joined at *P*, have the requisite inclination *K*, as well as the planes *efg*, *efp*. Continuing the comparison, in this way, by successive steps, it

Fig. 283.

will appear that the two surfaces adjust themselves perfectly to each other, and form a single continuous convex surface; which will be that of the regular dodecaedron, since it is composed of twelve equal regular pentagons, and has all its solid angles equal to each other.

Construction of the Icosaedron.

Fig. 284. 577. Let ABC (*fig.* 284) be one of its faces. We must first form a solid angle with five planes each equal to ABC, and each equally inclined to its adjacent one. To effect this, on the side $B'C'$, equal to BC, construct the regular pentagon $B'C'H'I'D'$; at the centre of this pentagon, draw a line at right angles to its plane, and terminating in A', so that $B'A' = B'C'$; join $A'C'$, $A'H'$, $A'I'$, $A'D'$; the solid angle A' formed by the five planes $B'A'C'$, $C'A'H'$, &c., will be the solid angle required. For the oblique lines $A'B'$, $A'C'$, &c. are equal; one of them $A'B'$ is equal to the side $B'C'$; hence all the triangles $B'A'C$, $C'A'H'$, &c. are equal to each other and to the given triangle ABC.

It is farther manifest, that the planes $B'A'C$, $C'A'H'$, &c., are all equally inclined to their adjacent planes; for the solid angles B', C', &c., are all equal to each other, being each formed by two angles of equilateral triangles, and one of a regular pentagon. Let K be the inclination of two planes, forming the equal angles, which inclination may be determined by art. 361; the angle K will at the same time be the inclination of each of the planes composing the solid angle A' to their adjacent planes.

This being granted, if at each of the points A, B, C, a solid angle be formed equal to the angle A', we shall have a convex surface $DEFG$, &c., composed of ten equilateral triangles, every one of which will be inclined to its adjacent triangle by the quantity K; and the angles D, E, F, &c., of its contour will alternately combine three angles and two angles of equilateral triangles. Conceive a second surface equal to the surface $DEFG$, &c.; these two surfaces will adapt themselves to each other, if each triple angle of the one is joined to each double angle of the other; and, since the planes of these angles have already the common inclination K, requisite to form a quintuple solid angle equal to the angle A, this junction will require no change in the state of either surface, and the two together will form a single continuous surface, composed of twenty equilateral triangles. This surface will be that of the regular icosaedron, since all its solid angles are equal to each other.†

PROBLEM.

578. *To find the inclination of two adjacent faces of a regular polyedron.*

Solution. This inclination is deduced immediately from the construction we have just given of the five regular polyedrons, taken in connexion with art. 361, by means of which the three plane angles that form a solid angle being given, the angle which two of these plane angles form with each other may be determined.

In the tetraedron. Each solid angle is formed of three angles of equilateral triangles; therefore seek, by the problem referred to, the angle which two of these planes contain between them, and it will be the inclination of two adjacent faces of the tetraedron.

In the hexaedron. The angle contained by two adjacent faces is a right angle.

In the octaedron. Form a solid angle with two angles of equilateral triangles and a right angle; the inclination of the two planes, in which the triangular angles are situated, will be that of two adjacent faces of the octaedron.

In the dodecaedron. Every solid angle is formed by three angles of regular pentagons; the inclination of the planes of two of these angles will be that of two adjacent faces of the dodecaedron.

In the icosaedron. Form a solid angle with two angles of equilateral triangles and one of a regular pentagon; the inclination of the two planes, in which the triangular angles are situated, will be that of two adjacent faces of the icosaedron.

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† If the figures 287, 288, 289, 290, 291, be accurately drawn on pasteboard and the fine lines be cut through, and the full lines cut only half through, the edges of the several polygons in each figure may be brought together and glued, the shaded one remaining fixed. Models of the several regular polyedrons may thus be easily obtained.

PROBLEM.

579. *The side of a regular polyedron being given, to find the radius of the inscribed and that of the circumscribed sphere.*

Solution. It must first be shown, that every regular polyedron is capable of being inscribed in a sphere, and of being circumscribed about it.

Fig. 292. Let *AB* (*fig.* 292) be the side common to two adjacent faces; *C* and *E* the centres of those faces; *CD*, *ED*, the perpendiculars let fall from these centres upon the common side *AB*, and therefore terminating in *D* the middle point of that side. The two perpendiculars *CD*, *DE*, make with each other an angle which is known, being the inclination of two adjacent faces, and determinable by the last problem. Now, if in the plane *CDE*, at right angles to *AB*, two indefinite lines *CO* and *OE* be drawn perpendicular to *CD* and *ED*, and meeting each other in *O*; this point *O* will be the centre of the inscribed and of the circumscribed sphere, the radius of the first being *OC*, that of the second *OA*.

For, since the perpendiculars *CD*, *DE*, are equal, and the hypothenuse *DO* is common, the right-angled triangle *CDO* must (56) be equal to the right-angled triangle *ODE*, and the perpendicular *OC* to *OE*. But, *AB* being perpendicular to the plane *CDE*, the plane *ABC* (349) is perpendicular to *CDE*, or *CDE* to *ABC*; likewise *CO*, in the plane *CDE* is perpendicular to *CD*, the common intersection of the planes *CDE*, *ABC*; hence (351) *CO* is perpendicular to the plane *ABC*. For the same reason, *EO* is perpendicular to the plane *ABE*; hence the two straight lines *CO*, *EO*, drawn perpendicular to the planes of two adjacent faces, through the centres of those faces, will meet in the same point *O*, and be equal to each other. Now, suppose that *ABC* and *ABE* represent any other two adjacent faces; the perpendicular *CD* will still continue of the same magnitude; and also the angle *CDO*, the half of *CDE*; consequently the right-angled triangle *CDO*, and its side *CO* will be equal in all the faces of the polyedron; hence, if from the point *O* as a centre with the radius *OC*, a sphere be described, it will touch all the faces of the polyedron at their centres, the planes *ABC*, *ABE*, &c., being each perpendicular to a radius at its extremity; therefore the sphere will be inscribed in the polyedron, or the polyedron circumscribed about the sphere.

Again, join *OA*, *OB*; since $CA = CB$, the two oblique lines *OA*, *OB*, being equally remote from the perpendicular, will be equal; so also will any other two lines drawn from the centre *O* to the extremities of any one side; hence all those lines will be equal to each other; and, if from the point *O* as a centre, with the radius *OA*, a spherical surface be described, it will pass through the vertices of all the solid angles of the polyedron; hence the sphere will be circumscribed about the polyedron, or the polyedron inscribed in the sphere.

This being settled, the solution of the problem presents no farther difficulty, and may be effected thus:

One face of the polyedron being given, describe that face; and let *CD* (*fig.* 293) be a perpendicular from its centre upon one of its sides. Find, by the last problem, the inclination of two adjacent faces of the polyedron, and make the angle *CDE* equal to this inclination; take $DE = CD$; draw *CO* and *EO* perpendicular to *CD* and *ED*, respectively; these two perpendiculars will meet in a point *O*; and *CO* will be the radius of the sphere inscribed in the polyedron. Fig. 293.

On the prolongation of *DC*, take *CA* equal to a radius of the circle, which circumscribes a face of the polyedron; *AO* will be the radius of the sphere circumscribed about this same polyedron.

For, the right-angled triangles *CDO*, *CAO*, in the present diagram, are equal to the triangles of the same name in the preceding diagram; and thus, while *CD* and *CA* are the radii of the inscribed and the circumscribed circles belonging to any one face of the polyedron, *OC* and *OA* are the radii of the inscribed and the circumscribed spheres which belong to the polyedron itself.

580. *Scholium.* From the foregoing propositions, several consequences may be deduced.

I. Any regular polyedron may be divided into as many regular pyramids as the polyedron has faces; the common vertex of these pyramids will be the centre of the polyedron; and at the same time, that of the inscribed and of the circumscribed sphere.

II. The solidity of a regular polyedron is equal to its surface multiplied by a third part of the radius of the inscribed sphere.

III. Two regular polyedrons of the same name are two similar solids, and their homologous dimensions are proportional; hence the radii of the inscribed or of the circumscribed spheres are to each other as the sides of the polyedrons.

IV. If a regular polyedron is inscribed in a sphere, the planes drawn from the centre, along the different edges, will divide the surface of the sphere into as many spherical polygons, as the polyedron has faces all equal and similar among themselves.

Improved Demonstration of the Theorem for the Solidity of the Triangular Pyramid.

BY M. QUERET OF ST. MALO.

THEOREM.

568. Two *triangular pyramids, having equivalent bases and equal altitudes, are equivalent, or equal in solidity.*

Let S-ABC, s-abc (*fig.* 294) be two triangular pyramids of Fig. 294. which the two bases ABC, abc, supposed to be situated in the same plane, are equivalent, the altitude TA being the same in both. If they are not equivalent, let s-abc, be the smaller; and suppose Aa to be the altitude of a prism, which having ABC for its base, is equal to their difference.

Divide the altitude AT into equal parts Ax, xy, yz, &c., each less than Aa, and let k be one of these parts; through the points of division suppose planes parallel to the plane of the bases; the corresponding sections formed by these planes in the two pyramids will be respectively equivalent by art. 409, namely, DEF, to def, GHI, to ghi, &c.

This being granted, upon the triangles ABC, DEF, GHI, &c., taken as bases, construct exterior prisms having for edges the parts AD, DG, GK, &c., of the side SA; in like manner, on the bases def, ghi, klm, &c., in the second pyramid, construct interior prisms having for edges the corresponding parts of sa. It is plain that the sum of all the exterior prisms of the pyramid S-ABC will be greater than this pyramid; and also that the sum of all the interior prisms of the little pyramid s-abc will be less than this. Hence the difference between the sum of all the exterior prisms and the sum of all the interior ones, must be greater than the difference between the two pyramids themselves.

Now, beginning with the bases *ABC*, *a b c*, the second exterior prism *DEFG* is equivalent to the first interior prism *d e f a*, because they have the same altitude *k*, and their bases *DEF*, *d e f*, are equivalent; for like reasons, the third exterior prism *GHIK* and the second interior prism *g h i d* are equivalent; the fourth exterior and the third interior; and so on, to the last in each series. Hence all the exterior prisms of the pyramids *S-ABC*, excepting the first prism *DABC*, have equivalent corresponding ones in the interior prisms of the pyramid *s-a b c*; hence the prism *DABC* is the difference between the sum of all the exterior prisms of the pyramid *S-ABC*; and the sum of all the interior prisms of the pyramid *s-a b c*. But the difference between these two sets of prisms has already been proved to be greater than that of the two pyramids; which latter difference we supposed to be equal to the prism *a ADC*; hence the prism *DABC* must be greater than the prism *a ABC*. But in reality it is less; for they have the same base *ABC*, and the altitude *A x* of the first is less than *A a* the altitude of the second. Consequently the supposed inequality between the two pyramids cannot exist; therefore the two pyramids *S-ABC*, *s a b c*, having equal altitudes and equivalent bases, are themselves equivalent.

THEOREM.

569. *Every triangular pyramid is a third part of the triangular prism having the same base and same altitude.*

Fig. 216. *Demonstration.* Let *F-ABC* (*fig.* 216) be a triangular pyramid, *ABCDEF* a triangular prism of the same base and the same altitude; the pyramid will be equal to a third of the prism.

Cut off the pyramid *F-ABC* from the prism, by a section made along the plane *FAC*; there will remain the solid *FACDE*, which may be considered as a quadrangular pyramid, whose vertex is *F*, and whose base is the parallelogram *ACDE*. Draw the diagonal *CE*; and extend the plane *FCE*, which will cut the quadrangular pyramid into two triangular ones *F-ACE* *F-CDE*. These two triangular pyramids have for their common altitude the perpendicular let fall from *F* on the plane *ACDE*; they have equal bases, the triangles *ACE*, *CDE*, being halves of the same parallelogram; hence (568) the two pyramids *F-ACE*, *F-CDE*, are equivalent. But the pyramid *F-CDE* and the pyramid *F-ABC* have equal bases *ABC*, *DEF*; they have also the same altitude, namely, the distance of the parallel planes *ABC*, *DEF*; hence the two pyramids are equivalent. Now the pyramid *F-CDE* has already been proved equivalent to *F-ACE*; consequently the three pyramids *F-ABC*, *F-CDE*, *F-ACE*, which compose the prism *ABD* are all equivalent. Therefore the pyramid *F-ABC* is the third part of the prism *ABD*, which has the same base and the same altitude.

570. *Corollary.* The solidity of a triangular pyramid is equal to a third part of the product of its base by its altitude.

NOTES.

I,

Upon certain Names and Definitions.

Some new expressions and definitions have been introduced into this work which tend to give to the language of geometry more exactness and precision. We proceed to give an account of these changes, and to propose certain others, which might fulfil more completely the same purposes.

In the ordinary definition of a *rectangular parallelogram* and of a *square*, it is said that the angles of these figures are right angles; it would be more exact to say, that their angles are equal. For, to suppose that the four angles of a quadrilateral may be right angles, and also that these right angles are equal to each other, is to suppose propositions which require to be demonstrated. This inconvenience and several others of the same kind might be avoided, if, instead of putting the definitions, as is usual, at the head of a section, we distributed them through the section each in the place where the proposition implied is demonstrated.

The word *parallelogram* according to its etymology signifies *parallel lines;* it answers not better to a figure of four sides than to one of six, eight, &c., the opposite sides of which are parallel. Likewise the word *parallelopiped* signifies *parallel planes;* it does not designate a solid of six faces any more than one of eight, ten, &c., of which the opposite ones are parallel. It seems then that the denominations of parallelogram and parallelopiped, which have besides the inconvenience of being very long, ought to be banished from geometry. We might substitute in their place those of *rhomb* and *rhomboid*, which are much more convenient, and preserve the name of *lozenge* to denote a quadrilateral, the sides of which are equal.

The word *inclination* ought to be understood in the same sense as that of angle; each indicates the manner of being of two lines, or of two planes, which meet, or which produced would meet. The inclination of two lines is nothing, when the angle is nothing, that is, when the lines are parallel or coincident. The inclination is greatest, when the angle is greatest, or when the two lines make with each other a very obtuse angle. The quality of *leaning* is taken in a different sense; a line *leans* so much the more with respect to another, as it departs more from a perpendicular to this last.

The denomination of *equal angles* is given by Euclid and others to those triangles, which are only equal in surface; and that of *equal solids* to those which are only equal in solidity. It appears to us more proper to call the triangles as well as the solids in this case *equivalent*, and to restrict the denomination of *equal triangles* and *equal solids* to those which would coincide upon being applied.

It is, moreover, necessary to distinguish among solids and curved surfaces two different kinds of equality. Indeed two solids, two solid angles, two spherical triangles, or two spherical polygons, may be equal in all their constituent parts without coinciding when applied. It does not appear that this observation has been made in elementary books; and, for want of having regard to it, certain demonstrations, founded upon the coincidence of figures, are not exact. Such are the demonstrations by which several authors pretend to prove the equality of spherical triangles in the same cases and in the same manner as they do that of plane triangles. We are furnished with a striking example of this by Robert Simson, who, in attacking the demonstration of the 28th proposition of the eleventh book of Euclid, fell himself into the error of founding his demonstration upon a coincidence which does not exist. We have thought it proper, therefore, to give a particular name to this kind of equality, which does not admit of coincidence; we have called it *equality by symmetry*; and the figures which are thus related we call *symmetrical* figures.

Thus the denominations of *equal* figures, *symmetrical* figures, *equivalent* figures, refer to different things and ought not to be confounded.

In the propositions, which relate to polygons, solid angles, and polyedrons, we have expressly excluded those which have re-entering angles. For, in addition to the advantage of considering in the elements only the most simple figures, if we had not thus restricted ourselves, certain propositions would either not have been true, or would have required to be modified. We have, therefore, confined ourselves to the consideration of lines and surfaces, which we call *convex*, and which are such that they cannot be cut by a straight line in more than two points.

We have often used the expression *product of two or of a greater number of lines*, by which we mean the product of the numbers which represent those lines, they being estimated according to a linear unit taken at pleasure. The sense of this word being thus fixed, there is no difficulty in making use of it. The same is to be understood of the product of a surface by a line, of a surface by a solid, &c. It is sufficient to have established once for all that these products are or ought to be considered as the products of numbers, each of a kind that is adapted to it. Thus the product of a surface by a solid is nothing else than the product of a number of superficial units by a number of solid units.

We often use the word *angle* in common discourse to designate the point situated at its vertex; this expression is faulty. It would be more clear and more exact to denote by a particular name, as that of *vertices*, the points situated at the vertices of the angles of a polygon, or of a polyedron. In this sense is to be understood the expression *vertices of a polyedron*, which we have used.

We have followed the common definition of *similar rectilineal figures;* but we would observe, that it contains three superfluous conditions. For, in order to construct a polygon of which the number of sides is n, it is necessary in the first place to know a side, and then to have the position of the vertices of the angles situated without this side. Now the number of these angles is $n - 2$, and the position of each vertex requires two data; whence it follows that the whole number of data necessary to construct a polygon of n sides is $1 + 2n - 4$, or $2n - 3$. But in the similar polygon there is one side to be taken at pleasure; thus the number of conditions, by which one polygon becomes similar to a given polygon, is $2n - 4$. But the common definition requires, 1. that the angles should be equal, each to each, which makes n conditions; 2. that the homologous sides should be proportional, which makes $n - 1$ conditions. There are then in all $2n - 1$ conditions, or three too many. In order to obviate this inconvenience we can resolve the definition into two others, in this manner.

1. *Two triangles are similar, when they have two angles equal, each to each.*

2. *Two polygons are similar, when there can be formed in the one and the other the same number of triangles similar, each to each, and similarly disposed.*

But, in order that this last definition should not itself contain superfluous conditions, it is necessary that the number of triangles should be equal to the number of sides of the polygon minus two, which may take place in two ways. We can draw from two homologous angles diagonals to the opposite angles; then all the triangles formed in each polygon will have a common vertex, and their sum will be equal to the polygon; or rather we can suppose that all the triangles formed in a polygon have for a common base a side of the polygon, and for vertices those of the different angles opposite to this base. In each case the number of triangles formed being $n - 2$, the conditions of their similitude will be equal to the number $2n - 4$; and the definition will contain nothing superfluous. This new definition being adopted, the ancient one will become a theorem, which may be demonstrated immediately.

If the definition of similar rectilineal figures is imperfect in books of elements, that of *similar solid polyedrons* is still more so. In Euclid this definition depends upon a theorem not demonstrated; in other authors it has the inconvenience of being very redundant; we have, therefore, rejected these definitions of similar solids.†

The definition of a *perpendicular to a plane* may be regarded as a theorem; that of the *inclination of two planes* also requires to be supported by reasoning; the same may be said of several others. It is on this account that, while we have placed the definitions according to ancient usage, we have taken care to refer to propositions where they are demonstrated; sometimes we have merely added a brief explanation which appeared sufficient.

The *angle* formed by the *meeting of two planes*, and the solid angle formed by the meeting of several planes in the same point, are distinct kinds of magnitudes, to which it would be well perhaps to give particular names. Without this it is difficult to avoid obscurity and circumlocutions in speaking of the arrangement of planes which compose the surface of a polyedron; and as the theory of solids has been little cultivated hitherto, there is less inconvenience in introducing new expressions, where they are required by the nature of the subject.

I should propose to give the name of *wedge* to the angle formed by two planes; the *edge* or *height* of the wedge would be the common intersection of the two planes. The wedge would be designated by four letters, of which the two middle ones would answer to the edge. A *right wedge* then would be the angle formed by two planes perpendicular to each other. Four right wedges would fill all the solid angular space about a given line. This new denomination would not prevent the wedge always having for its measure the angle formed by two lines drawn from the same point, the one in one of the planes and the other in the other, perpendicularly to the edge or common intersection.

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† The author here refers to a distinct note on the equality and similitude of polyedrons, not given in this translation.

II.

By the Translator.

The improvements referred to in the preceding note, so far as they have been adopted by the author, have been carefully preserved in the translation. Indeed it has been found necessary in a few instances to use English words in a sense somewhat different from their ordinary acceptation. The word *polygon* is generally restricted to figures of more than four sides. It is used in this work with the latitude of the original word *polygone* to stand for rectilineal figures generally; and *polyedron* is adopted in a similar manner for solids. *Quadrilateral* is employed as a general name for four-sided figures. The word *losenge* is rendered by *rhombus*, and *trapèze* by *trapezoid*, the English words, as they are commonly used, corresponding to the French. The perpendicular let fall from the centre of a regular polygon upon one of its sides is called in the original *apothême*. It occurs but a few times, and as there is no English word answering to it, it is rendered by a periphrasis, or simply by the word *perpendicular*. The portion of the surface of a sphere comprehended between the semicircumferences of two great circles is denoted in the original by *fuseau;* Dr. Hutton uses the word *lune* in the same sense; others have employed *lunary surface;* as lune properly stands for the surface comprehended between two unequal circular curves, the latter denomination was thought the least exceptionable, and is adopted in the translation.

III.

On the Demonstration of the Proposition of Article 58.

The proposition of art. 58 is only a particular case of the celebrated *postulate* upon which Euclid has established the theory of parallel lines, as well as the theorem upon the sum of the three angles of a triangle. This *postulate* has not yet been demonstrated in a manner entirely geometrical, and independent of the consideration of infinity, which is undoubtedly to be attributed to the imperfection of the definition of a straight line, which serves as the basis of the elements. But, if we consider this subject in a point of view more abstract, analysis offers a very simple method of demonstrating the proposition rigorously.

We show immediately by superposition, and without any preliminary proposition, that *two triangles are equal, when a side and the two adjacent angles of the one are equal to a side and the two adjacent angles of the other, each to each.* Let us call p the side in question, A and B the two adjacent angles, C the third angle. The angle C then must be entirely determinate, when the angles A and B are known with the side p; for, if several angles C could correspond to the three given things A, B, p, there would be as many different triangles, which would have a side and the two adjacent angles of the one equal to a side and the two adjacent angles of the other, which is impossible; therefore the angle C must be a determinate function of the three quantities A, B, p; which may be expressed thus

$$C = \varphi : (A, B, p).$$

Let the right angle be equal to unity, then the angles A, B, C, will be numbers comprehended between 0 and 2; and, since

$$C = \varphi\ (A, B, p),$$

we say that the line p does not enter into the function φ. Indeed we have seen that C must be entirely determined by the data A, B, p, merely, without any other angle or line whatever; but the line p is of a nature heterogeneous to the numbers A, B, C; and if, having any equation whatever among A, B, C, p, we could deduce the value of p, in A, B, C, it would follow that p is equal to a number, which is absurd; therefore p cannot enter into the function φ, and we have simply $C = \varphi : (A, B)$.....*.

This formula proves already that, if two angles of a triangle are equal to two angles of another triangle, the third must be equal to the third; and, this being supposed, it is easy to arrive at the theorem we have in view.

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* It has been objected to this demonstration that if it were applied, word for word, to spherical triangles, it would follow that two known angles would be sufficient to determine the third, which would not be true in this kind of triangles. The answer is, that in spherical triangles there is one element more than in plane triangles, and this element is the radius of the sphere which must not be omitted. Accordingly, let r be the radius; then, instead of having $C = \varphi\ (A, B, p)$, we shall have $C = \varphi\ (A, B, p, r)$, or simply $C = \varphi \left(A, B, \frac{p}{r}\right)$, by the law of homogenous quantities. Now, since the ratio $\frac{p}{r}$ is a number, as well as A, B, C, there is nothing to prevent $\frac{p}{r}$ being found in the fraction φ, and then we can no longer conclude that

$$C = \varphi\ (A, B).$$

In the first place let *ABC* (*fig.* 274) be a triangle right-angled at *A*; from the point *A* let fall upon the hypothenuse the perpendicular *AD*. The angles *B* and *D* of the triangle *ABD* are equal to the angles *B* and *A* of the triangle *BAC*; therefore, according to what has just been demonstrated, the third angle *BAD* is equal to the third *C*; for the same reason the angle $DAC = B$; consequently $BAD + DAC$, or $BAC = B + C$; but the angle *BAC* is a right angle; therefore *the two acute angles of a right-angled triangle, taken together, are equal to a right angle.* Fig. 274.

Again, let *BAC* (*fig.* 275) be any triangle, and *BC* a side which is not less than each of the two others; if from the opposite angle *A* the perpendicular *AD* be let fall upon *BC*, this perpendicular will fall within the triangle *ABC*, and will divide it into two right-angled triangles *BAD*, *DAC*. Now in the right-angled triangle *BAD* the two angles *BAD*, *ABD*, are together equal to a right angle; in the right-angled triangle *DAC* the two angles *DAC*, *ACD*, are also equal to a right angle. Consequently the four united, or the three *BAC*, *ABC*, *ACB*, are together equal to two right angles; therefore, *in every triangle the sum of the three angles is equal to two right angles.* Fig. 275.

We see by this that the theorem, considered *a priori*, does not depend upon a series of propositions, but is deduced immediately from the principle of homogeneity, a principle which exists in every relation among quantities of whatever kind. But we proceed to show that another fundamental theorem of geometry may be deduced from the same source.

The above denominations being preserved, and the side opposite to the angle *A* being called m, and the side opposite the angle *B* being called n; the quantity m must be entirely determined by the quantities A, B, p; consequently m is a function of A, B, p, as also $\frac{m}{p}$, so that we can make $\frac{m}{p} = \psi : (A, B, p)$. But $\frac{m}{p}$ is a number, as well as A and B; therefore the function ψ must not contain the line p, and we have simply $\frac{m}{p} = \psi : (A, B)$, or $m = p\,\psi : (A, B)$. We have also in a similar manner $n = p\,\psi : (A, B)$.

Let there be another triangle formed with the same angles A, B, C, and having for the opposite sides m', n', p', respectively. Since A and B do not change, we have in this new triangle

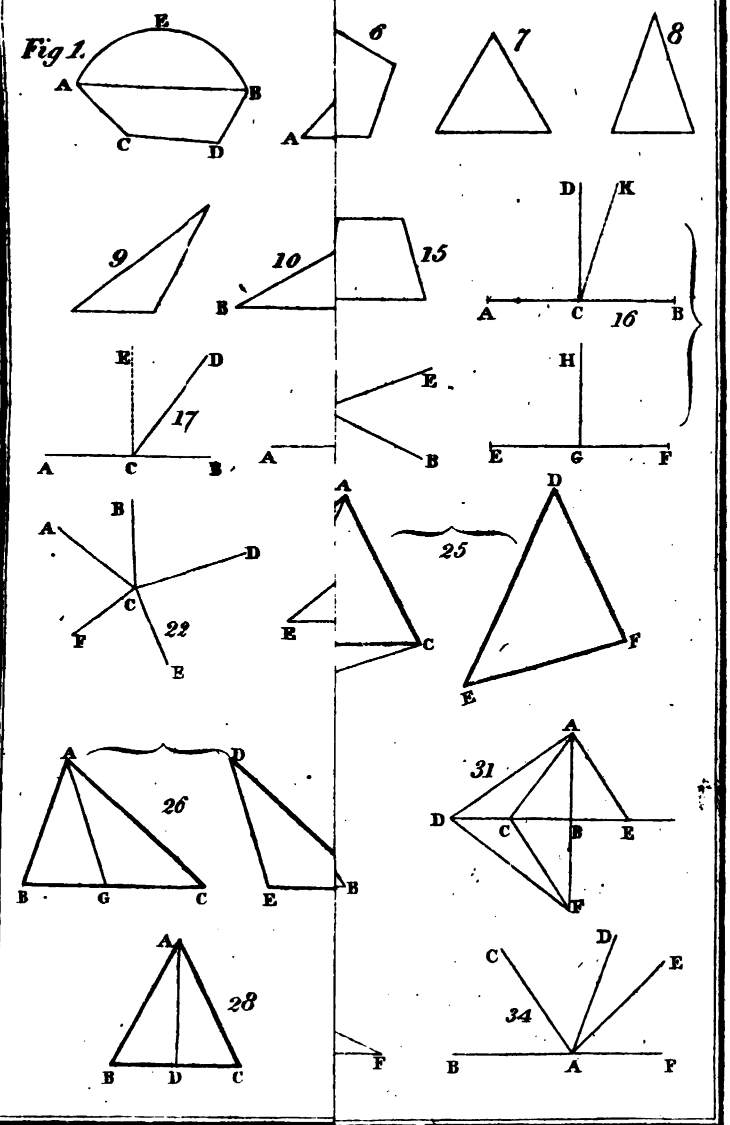
$$m' = p'\,\psi\,(A, B),$$

and $n' = p'\,\psi : (B, A)$. Therefore $m : m' :: n' :: p : p'$. Therefore *in equiangular triangles the sides opposite to the equal angles are proportional.*

From this general proposition we deduce, as a particular case, that which we have supposed in the text for the demonstration of the Fig. 35. proposition of art. 58. Indeed the triangles, *AFG*, *AML* (*fig* 35), have two angles equal, each to each, namely, the angle *A* common, and a right angle. Consequently these triangles are equiangular; therefore we have the proportion *AF* : *AL* : : *AG* : *AM*, by means of which the proposition is fully demonstrated[†].

[†] In the note, of which the above is only a part, the author undertakes to demonstrate in a similar manner other fundamental propositions of geometry. For remarks upon the kind of reasoning here employed, the reader is referred to Leslie's Geometry, third edition, page 292.

Cambridge Mathematics. Elements of Geometry. *Pl. 1.*



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